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# Singular Perturbation of Quantum Stochastic Differential Equations with Coupling Through an Oscillator Mode[1]

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We consider a physical system which is coupled indirectly to a Markovian resevoir through an oscillator mode. This is the case, for example, in the usual model of an atomic sample in a leaky optical cavity which is ubiquitous in quantum optics. In the strong coupling limit the oscillator can be eliminated entirely from the model, leaving an effective direct coupling between the system an the resevoir. Here we provide a mathematically rigorous treatment of this limit as a weak limit of the time evolution and observables on a suitably chosen exponential domain in Fock space. The resulting effective model may contain emission and absorption as well as scattering interactions.

**KEY WORDS:** singular perturbation, quantum stochastic differential equations, Hudson–Parthasarathy quantum stochastic calculus, adiabatic elimination

## 1. INTRODUCTION

The motivation for this article stems from the following problem in quantum optics, illustrated in Fig. 1. Consider the canonical starting point of cavity QED, an atomic system in an optical cavity. In many cases such a system is well modelled using only a single cavity mode. The cavity is then effectively described by a single quantum harmonic oscillator, and the atom-cavity interaction Hamiltonian takes the form

$$H = E_{11}b^\dagger b + E_{10}b^\dagger + E_{01}b + E_{00}, \tag{1}$$

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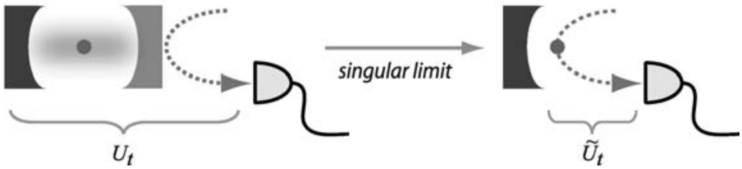


**Fig. 1.** (color online) Cartoon illustration of a prototypical problem that falls within the framework of this paper. An atom is placed in a single-mode optical cavity. One of the cavity mirrors is leaky, thus coupling the cavity mode to the external field (which may ultimately be detected). In the "bad cavity" limit the mirror is made so transmissive that the cavity can be eliminated from the description of the model, leaving an effective direct interaction between the atom and the field.

where $E_{ij}$ are operators acting on the atomic Hilbert space, $E_{ij}^\dagger = E_{ji}$, and $b$, $b^\dagger$ are the cavity mode annihilation and creation operators, respectively. Usually one of the cavity mirrors is assumed to be perfectly reflective, while the other mirror allows some light to leak into the electromagnetic field outside the cavity and vice versa. In the Markov approximation,[1,2] the time evolution of the entire system (consisting of the atom, cavity and external field) is described by the unitary solution to the Hudson–Parthasarathy[16] quantum stochastic differential equation

$$dU_t = \left\{ \sqrt{\gamma}\, b dA_t^\dagger - \sqrt{\gamma}\, b^\dagger\, dA_t - \frac{\gamma}{2}\, b^\dagger b\, dt - iH\, dt \right\} U_t, \tag{2}$$

where $A_t$, $A_t^\dagger$ are the usual creation and annihilation processes in the external field. The transmissivity of the leaky mirror is controlled by the positive constant $\gamma$. The physical origin of this model is briefly elaborated on in appendix A.

In many situations of practical interest, $\gamma$ will be quite large compared to the strengths $\|E_{ij}\|$ of the atom-cavity interaction. When this is the case, one would expect that the presence of the cavity has little qualitative influence on the atomic dynamics: the cavity is then essentially transparent in the frequency range corresponding to the atomic dynamics, so that the atoms "see" the external field directly. Similarly, we expect that measurements obtained from detection of the outgoing field (e.g., by homodyne detection) would depend directly on the atomic observables and would be essentially independent of the cavity observables. The hope is, then, that the time evolution $U_t$ can be described in some idealized limit by the unitary solution $\tilde{U}_t$ of a new Hudson–Parthasarathy equation which involves only atomic operators and the external field, and in which the cavity has been eliminated. The goal of this article is to make these ideas precise.

## 1.1. Previous Work

The elimination of a leaky cavity in the bad cavity limit is an extremely common procedure in the physics literature—so common, in fact, that most papers state the resulting expression without further comment (*"we adiabatically eliminate the cavity, giving ..."*). Often the equation considered is a Lindblad-type master equation for the atom and cavity; in our context, this (deterministic) differential equation for the reduced density operator can be obtained by averaging over the field as in Ref. 16. One method that is used to eliminate the cavity in such an equation, see e.g. Ref. 26, involves expanding the density operator in matrix elements, setting certain time derivatives to zero, then solving algebraically to obtain an equation for the atomic matrix elements only. This method is commonly known as adiabatic elimination. Though such an approach is not very rigorous, similar techniques can sometimes be justified in the context of the classical theory of singular perturbations (Tikhonov's theorem[23]). A somewhat different approach, see e.g. Ref. 11, uses projection operators and Laplace transform techniques. None of these techniques are applicable to the question posed here, however, as we wish to retain the external field in the limiting model. Hence we are seeking a singular perturbation result for quantum stochastic differential equations, which is (to our knowledge) not yet available in the literature.

A naive attempt at adiabatic elimination for quantum stochastic equations is made in Ref. 7 (see also[4] Refs. 8, 27). These authors use the following procedure:

- First, they obtain Heisenberg equations of motion (in Itô form) for the cavity annihilator $b_t = U_t^\dagger b U_t$ and also for the relevant atomic operators.
- Next, they set $\dot{b}_t = 0$ (where the right-hand side is interpreted as "quantum white noise") and solve algebraically for $b_t$.
- Next, they plug this expression into the atomic equations of motion.
- Finally, they interpret these equations as "implicit" equations[25] (a formal analog of Stratonovich equations) and convert to the "explicit form" (a formal analog of Itô equations). The latter are considered to be the adiabatically eliminated Heisenberg equations of motion for the atomic operators.

Attempts at justifying this procedure run into a number of seemingly fatal problems. Forgoing the issue of the mathematical well-posedness of "quantum white noise" and the fact that $\dot{b}_t = 0$ seems incompatible with the fact that the right-hand side is formally infinite, it is unclear how the resulting equation should be interpreted. Even in the classical stochastic case, it is known that adiabatically eliminated expressions need not be of Stratonovich type (see Ref. 10 for some counterexamples); the singular limit is rather delicate and the resulting outcome depends on the way in which the limit is taken. Besides, it should be pointed out

[4] We also mention[24] where some results of Ref. 7 are reconsidered. The results are only reproduced, however, at the master equation level; in particular, the quantum noise is not retained and the implicit-explicit formalism is not used in those sections where results of Ref. 7 are considered.

that the implicit-explicit formalism introduced in Ref. 25 (essentially along the lines of McShane's canonical extension[(17,18)]) does not even capture correctly the ordinary quantum Markov limit in the presence of scattering interactions; compare the expressions in Ref. 25 to the rigorous results obtained in Ref. 14. Finally, there are serious issues with operator ordering, arising from the fact that the exact $b_t$ and the "slaved" $b_t$ that is obtained by solving $\dot{b}_t = 0$ do not obey the same commutation relations. Some further details are provided in Appendix A, where we show that an operator ordering and interpretation (which are different than those used in Ref. 7) can be chosen so that such a formal procedure gives the right answer. From the outset, however, these choices are no more plausible than any other, which highlights the necessity of a careful and more rigorous analysis.[5]

## 1.2. Statement of the Problem

Following Ref. 10, we seek a *"method by which fast variables may be eliminated from the equations of motion in some well-defined limit."* Which limit to take is not entirely obvious at the outset; for example, the naive choice $\gamma \to \infty$ only yields trivial results (the cavity is forced to its ground state and the atomic dynamics vanishes). To define a nontrivial limit, we introduce the scaling parameter $\varepsilon > 0$ and make the substitution $b \mapsto \varepsilon^{-1/2} b$ in (1) and (2). The limit $\varepsilon \to 0^+$ then has the character of a central limit theorem, and provides a nontrivial result in which the cavity is eliminated. (Note that similar scaling limits are used in projection operator techniques for master equations.[(11)])

Our approach, then, is to proceed as follows. First we make the above substitution. Next we switch to the interaction picture with respect to the cavity-field interaction. This gives rise to an interaction picture time evolution in which the atom is driven by a quantum Ornstein–Uhlenbeck process. The limit $\varepsilon \to 0^+$ corresponds essentially to a Markov limit of this equation, and consequently our proofs borrow heavily from the methods developed to treat such limits (particularly from the estimates developed in Ref. 14). However, our limits are of a somewhat stronger character than those considered in Refs. 1, 2, 14 as we take weak limits on a fixed domain in the underlying Hilbert space, rather than "limits in matrix elements" where the domain depends on $\varepsilon$. We also consider, aside from the time evolution unitary and the Heisenberg evolution of the atomic observables, the limiting behavior of the output field operators (which can be observed e.g. through homodyne detection).

For concreteness, we will restrict ourselves to the model described by (1) and (2). This model is already very rich and widely used in the literature in

[5] Despite the use of an incorrect procedure, the authors of Ref. 7 succeed in obtaining the correct result for their model (see Example 1 in Sec. 4) due to a miraculous cancellation of errors, which is however a coincidence specific to their particular model. See Appendix A for further remarks.

various scenarios. Our results can also be extended to more complicated setups, in particular to the case of multiple external fields and oscillators along the lines of Ref. 15; the subsequent extension to thermal and squeezed noises is then also straightforward through the usual double Fock space construction, see e.g. Ref. 13, at least in the absence of scattering interactions ($E_{11} = 0$).

## 2. PRELIMINARIES

Throughout this article we work on the product Hilbert space $\mathfrak{h} = \mathfrak{h}_{\mathrm{sys}} \otimes \mathfrak{h}_{\mathrm{osc}} \otimes \mathfrak{h}_{\mathrm{resv}}$ consisting of a physical (e.g. atomic) system $\mathfrak{h}_{\mathrm{sys}}$, a quantum harmonic oscillator $\mathfrak{h}_{\mathrm{osc}} = \Gamma(\mathbb{C})$ (describing e.g. a cavity mode), and an external Bosonic resevoir $\mathfrak{h}_{\mathrm{resv}} = \Gamma(L^2(\mathbb{R}_+))$ (describing e.g. the electromagnetic field). Here $\Gamma(\mathfrak{h}')$ denotes the symmetric (Boson) Fock space over the one-particle Hilbert space $\mathfrak{h}'$. We use the following notation for Fock space vectors: $|0\rangle \in \Gamma(\mathfrak{h}')$ denotes the vacuum vector, $|f\rangle \in \Gamma(\mathfrak{h}')$ denotes the exponential vector corresponding to $f \in \mathfrak{h}'$, and $\mathcal{E} \subset \Gamma(\mathfrak{h}')$ denotes the linear space generated by the exponential vectors (the exponential domain). We will also use the subscripts $|f\rangle_{\mathrm{osc}}$ or $|f\rangle_{\mathrm{resv}}$, and similarly $\mathcal{E}_{\mathrm{osc}}$, $\mathcal{E}_{\mathrm{resv}}$, wherever confusion may arise.

We define the following standard operators: $b$ and $b^\dagger$ are the creation and annihilation operators on $\mathfrak{h}_{\mathrm{osc}}$, and $A_t$, $A_t^\dagger$ and $\Lambda_t$ are[6] the usual annihilation, creation and gauge processes on $\mathfrak{h}_{\mathrm{resv}}$, respectively.[(16)] We denote the ampliations of these operators to $\mathfrak{h}$ by the same symbols. For any $f \in L^2(\mathbb{R}_+)$ and for any real, bounded $g \in L^\infty(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ we also define the field operators[(16)]

$$A(f) = \int_0^\infty f(t)^* \, dA_t, \quad A(f)^\dagger = \int_0^\infty f(t) \, dA_t^\dagger, \quad \Lambda(g) = \int_0^\infty g(t) \, d\Lambda_t.$$

We recall that the exponential domain $\mathcal{E}_{\mathrm{resv}}$ can be extended to $\mathcal{E}'_{\mathrm{resv}} \supset \mathcal{E}_{\mathrm{resv}}$ in such a way that $\mathcal{E}'_{\mathrm{resv}}$ is invariant under the action of $A(f)$, $A(f)^\dagger$ and $\Lambda(g)$, see e.g. Ref. 19, pp. 61–65. Similarly $\mathcal{E}_{\mathrm{osc}}$ can be extended to $\mathcal{E}'_{\mathrm{osc}}$ so that the latter is invariant under $b$, $b^\dagger$ and $b^\dagger b$. This means in particular that the domain $\mathcal{D}' = \mathfrak{h}_{\mathrm{sys}} \underline{\otimes} \mathcal{E}'_{\mathrm{osc}} \underline{\otimes} \mathcal{E}'_{\mathrm{resv}} \subset \mathfrak{h}$ is invariant under finite linear combinations of operators of the form $E \otimes \{b, b^\dagger, b^\dagger b\} \otimes \{A(f), A(f)^\dagger, \Lambda(g)\}$, and that commutators of such operators are well defined on $\mathcal{D}'$. Here $E$ is any bounded operator on $\mathfrak{h}_{\mathrm{sys}}$ and $\underline{\otimes}$ denotes the algebraic tensor product. We recall also the useful identities $|\alpha\rangle_{\mathrm{osc}} = \exp(\alpha b^\dagger)|0\rangle_{\mathrm{osc}}$ and $|f\rangle_{\mathrm{resv}} = \exp(A(f)^\dagger)|0\rangle_{\mathrm{resv}}$ (Ref. 20, Sec. II.20).

The starting point for our investigation is the rescaled version of (1) and (2). We consider the unitary solution $U_t(\varepsilon)$, given the initial condition $U_0(\varepsilon) = I$, to

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[6] See Appendix A for a heuristic description in terms of white noise.

the Hudson–Parthasarathy quantum stochastic differential equation (QSDE)

$$dU_t(\varepsilon) = \left\{ \sqrt{\frac{\gamma}{\varepsilon}}\, b dA_t^\dagger - \sqrt{\frac{\gamma}{\varepsilon}}\, b^\dagger\, dA_t - \frac{\gamma}{2\varepsilon}\, b^\dagger b\, dt - i H(\varepsilon)\, dt \right\} U_t(\varepsilon), \tag{3}$$

where the Hamiltonian $H(\varepsilon)$ is taken to be of the form

$$H(\varepsilon) = \frac{1}{\varepsilon}\, E_{11} b^\dagger b + \frac{1}{\sqrt{\varepsilon}}\, E_{10} b^\dagger + \frac{1}{\sqrt{\varepsilon}}\, E_{01} b + E_{00}. \tag{4}$$

Here $E_{ij}$, $\|E_{ij}\| < \infty$ are (the ampliations of) given bounded operators on $\mathfrak{h}_{\rm sys}$, $E_{ij}^\dagger = E_{ji}$, and $\gamma, \varepsilon > 0$ are positive constants. The existence, uniqueness and unitarity of the solution of (3) are established in Ref. 9.

## 2.1. The Interaction Picture

We are interested in the limit $\varepsilon \to 0^+$. As the oscillator-resevoir dynamics becomes singular in this limit, the first step we take is to remove these dynamics by going over to the interaction representation. To this end, define the oscillator-resevoir time evolution $V_t(\varepsilon)$ as the unitary solution of

$$dV_t(\varepsilon) = \left\{ \sqrt{\frac{\gamma}{\varepsilon}}\, b\, dA_t^\dagger - \sqrt{\frac{\gamma}{\varepsilon}}\, b^\dagger\, dA_t - \frac{\gamma}{2\varepsilon}\, b^\dagger b\, dt \right\} V_t(\varepsilon),$$

where $V_0(\varepsilon) = I$. Existence, uniqueness and unitarity are again guaranteed by Ref. 9. We wish to consider the unitary

$$\tilde{U}_t(\varepsilon) = V_t(\varepsilon)^\dagger U_t(\varepsilon). \tag{5}$$

Using the quantum Itô rules,[16] we find that $\tilde{U}_t(\varepsilon)$ is given by the solution of the Schrödinger equation

$$\frac{d\tilde{U}_t(\varepsilon)}{dt} = -i\, \tilde{\Upsilon}_t(\varepsilon) \tilde{U}_t(\varepsilon), \quad \tilde{U}_0(\varepsilon) = I, \tag{6}$$

with time-dependent interaction Hamiltonian

$$\tilde{\Upsilon}_t(\varepsilon) = V_t(\varepsilon)^\dagger H(\varepsilon) V_t(\varepsilon). \tag{7}$$

Note that $V_t(\varepsilon)$ commutes with any system operator $E$ (on $\mathfrak{h}_{\rm sys}$), so we have $E_t(\varepsilon) = U_t(\varepsilon)^\dagger E U_t(\varepsilon) = \tilde{U}_t(\varepsilon)^\dagger E \tilde{U}_t(\varepsilon)$. Hence in order to study limits of the form $\lim_{\varepsilon \to 0^+} E_t(\varepsilon)$ it is sufficient to consider $\tilde{U}_t(\varepsilon)$ rather than $U_t(\varepsilon)$.

## 2.2. Quantum Ornstein–Uhlenbeck Processes

It is convenient to introduce the quantum Ornstein–Uhlenbeck (O–U) annihilation and creation processes

$$\tilde{a}_t(\varepsilon) = -\sqrt{\frac{\gamma}{4\varepsilon}}\, V_t(\varepsilon)^\dagger b V_t(\varepsilon), \quad \tilde{a}_t(\varepsilon)^\dagger = -\sqrt{\frac{\gamma}{4\varepsilon}}\, V_t(\varepsilon)^\dagger b^\dagger V_t(\varepsilon). \tag{8}$$

Using the quantum Itô rules, we find that

$$d\tilde{a}_t(\varepsilon) = \frac{\gamma}{2\varepsilon}\, dA_t - \frac{\gamma}{2\varepsilon}\, \tilde{a}_t(\varepsilon)\, dt, \quad \tilde{a}_0(\varepsilon) = -\sqrt{\frac{\gamma}{4\varepsilon}}\, b.$$

Solving explicitly, we obtain

$$\tilde{a}_t(\varepsilon) = \sqrt{\frac{\gamma}{4\varepsilon}}\, e^{-\gamma t/2\varepsilon} \left[ \sqrt{\frac{\gamma}{\varepsilon}} \int_0^t e^{\gamma u/2\varepsilon}\, dA_u - b \right]. \tag{9}$$

This allows us to express the interaction Hamiltonian $\tilde{\Upsilon}_t(\varepsilon)$ in the form

$$\tilde{\Upsilon}_t(\varepsilon) = \frac{4}{\gamma}\, E_{11} \tilde{a}_t(\varepsilon)^\dagger \tilde{a}_t(\varepsilon) - \frac{2}{\sqrt{\gamma}}\, E_{10} \tilde{a}_t(\varepsilon)^\dagger - \frac{2}{\sqrt{\gamma}}\, E_{01} \tilde{a}_t(\varepsilon) + E_{00}. \tag{10}$$

**Lemma 1.** *The O–U processes satisfy (on $\mathcal{D}'$) the commutation relations*

$$[\tilde{a}_t(\varepsilon), \tilde{a}_s(\varepsilon)] = 0, \quad [\tilde{a}_t(\varepsilon), \tilde{a}_s(\varepsilon)^\dagger] = G_\varepsilon(t-s), \tag{11}$$

*where the correlation function $G_\varepsilon(\tau)$ is given by*

$$G_\varepsilon(\tau) = \frac{\gamma}{4\varepsilon}\, \exp\left( -\frac{\gamma|\tau|}{2\varepsilon} \right).$$

**Proof:** Recall (Ref. 20, Sec. II.20) the commutation relation on $\mathcal{D}'$

$$[A(f), A(g)^\dagger] = \int_0^\infty f(t)^* g(t)\, dt.$$

Hence we obtain

$$\left[ e^{-\gamma t/2\varepsilon} \int_0^t e^{\gamma u/2\varepsilon}\, dA_u,\, e^{-\gamma s/2\varepsilon} \int_0^s e^{\gamma u/2\varepsilon}\, dA_u^\dagger \right]$$
$$= e^{-\gamma(t+s)/2\varepsilon} \int_0^{t\wedge s} e^{\gamma u/\varepsilon} du$$
$$= \frac{\varepsilon}{\gamma}\, e^{-\gamma(t+s-2\,t\wedge s)/2\varepsilon} - \frac{\varepsilon}{\gamma}\, e^{-\gamma(t+s)/2\varepsilon},$$

where $t \wedge s = \min(t, s)$. Now note that $t + s - 2\,t \wedge s = |t - s|$. Hence writing out the full commutators and using $[b, b^\dagger] = 1$, the result follows. □

The function $G_\varepsilon(\cdot)$ has the property of being strictly positive, symmetric and integrable with $\int_{-\infty}^{\infty} G_\varepsilon(\tau)\, d\tau = 1$. In the limit $\varepsilon \to 0^+$, $G_\varepsilon(\cdot)$ therefore converges in the sense of distributions to a delta function at the origin. This means that the Ornstein–Uhlenbeck processes $\tilde{a}_t(\varepsilon)$, $\tilde{a}_t(\varepsilon)^\dagger$ formally converge to quantum white noises as $\varepsilon \to 0^+$. Let us remark that these processes may now be written as

$$\tilde{a}_t(\varepsilon) = 2 \int_0^t G_\varepsilon(t-s)\, dA_s - \sqrt{\frac{4\varepsilon}{\gamma}}\, G_\varepsilon(t)\, b. \tag{12}$$

## 3. STRONG COUPLING LIMIT($\varepsilon \to 0^+$)

For any $g \in L^2(\mathbb{R}_+)$ we define the future and past smoothed functions

$$g^+(t,\varepsilon) = 2\int_0^\infty g(t+\tau) G_\varepsilon(\tau)\, d\tau = 2\int_t^\infty g(\tau) G_\varepsilon(t-\tau)\, d\tau,$$

$$g^-(t,\varepsilon) = 2\int_0^t g(t-\tau) G_\varepsilon(\tau)\, d\tau = 2\int_0^t g(\tau) G_\varepsilon(t-\tau)\, d\tau.$$

We will encounter such functions repeatedly in the following. If $g$ were a continuous function, we would have the limits

$$\lim_{\varepsilon \to 0^+} g^\pm(t,\varepsilon) = g(t). \tag{13}$$

The space $L^2(\mathbb{R}_+)$ is much too large, however, to ensure that the $\varepsilon \to 0^+$ limits of the smoothed functions are well behaved; consider for example a square integrable function with oscillatory discontinuity (e.g. $\sin(1/x)$). To avoid such unpleasantness we will restrict our attention to the set of regulated square integrable functions, following Ref. 12.

*Definition 1.* Let $L^2_\pm(\mathbb{R}_+) \subset L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ denote the set of square integrable bounded functions $f$ on the halfline such that the limits $f(t^\pm) = \lim_{s \to t^\pm} f(s)$ exist at every point $t \in \mathbb{R}_+$. We denote by $\mathcal{E}^\pm_{\text{resv}} \subset \mathcal{E}_{\text{resv}}$ the restricted exponential domain generated by exponential vectors with amplitude functions in $L^2_\pm(\mathbb{R}_+)$.

Before moving on, we make the following remarks:

1. Any $g \in L^2_\pm(\mathbb{R}_+)$ has at most a countable number of discontinuity points (see e.g. Ref. 5, chapter 3). Hence for such $g$ Eq. (13) holds for (Lebesgue-) a.e. $t \in \mathbb{R}_+$.
2. Note that if $g \in L^2_\pm(\mathbb{R}_+)$, then $\chi_{[0,s]} g \in L^2_\pm(\mathbb{R}_+)$ for any $s \in \mathbb{R}_+$. Hence $\mathcal{E}^\pm_{\text{resv}}$ is a suitable choice for the restricted exponential domain used

in the construction of the Hudson–Parthasarathy stochastic integration theory.[16]

For future reference, we collect various $\varepsilon \to 0^+$ limits in the following lemma.

**Lemma 2.** *For any $g \in L^2_\pm(\mathbb{R}_+)$, the following hold:* $g^\pm(t,\varepsilon) \xrightarrow{\varepsilon\to 0^+} g(t^\pm)$,

$$\lim_{\varepsilon\to 0^+}\int_0^\infty G_\varepsilon(t-s)g(s)\,ds = \lim_{\varepsilon\to 0^+} 2\int_0^t G_\varepsilon(t-s)g^+(s,\varepsilon)\,ds = \frac{g(t^+)+g(t^-)}{2}.$$

*Moreover, all these expressions are equal to $g(t)$ for (Lebesgue-)a.e. $t \in \mathbb{R}_+$; hence it follows that $g^\pm(\cdot,\varepsilon) \underset{\varepsilon\to 0^+}{\longrightarrow} g(\cdot)$ in $L^2(\mathbb{R}_+)$, etc.*

**Proof:** The first statement follows from

$$g^+(t,\varepsilon) = \int_0^\infty \frac{\gamma}{2}\, e^{-\gamma\tau/2}\, g(t+\varepsilon\tau)\,d\tau \xrightarrow{\varepsilon\to 0^+} g(t^+),$$

where we have used dominated convergence to take the limit. Similarly

$$g^-(t,\varepsilon) = \int_0^\infty \frac{\gamma}{2}\, e^{-\gamma\tau/2}\, g(t-\varepsilon\tau)\chi_{[0,t/\varepsilon]}(\tau)\,d\tau \xrightarrow{\varepsilon\to 0^+} g(t^-).$$

The third statement follows directly as

$$\int_0^\infty G_\varepsilon(t-s)g(s)\,ds = \frac{g^+(t,\varepsilon)+g^-(t,\varepsilon)}{2}.$$

To prove the next statement, note that

$$2\int_0^t G_\varepsilon(t-s)g^+(s,\varepsilon)\,ds = \int_0^\infty g(\tau)\left[4\int_0^{t\wedge\tau} G_\varepsilon(t-s)G_\varepsilon(s-\tau)\,ds\right]d\tau.$$

But straightforward calculation yields

$$4\int_0^{t\wedge\tau} G_\varepsilon(t-s)G_\varepsilon(s-\tau)\,ds = G_\varepsilon(t-\tau) - G_\varepsilon(\tau)\,\exp\left(-\frac{\gamma t}{2\varepsilon}\right),$$

and the result follows directly. Finally, the last statement of the lemma follows from the fact that $g$ has at most a countable number of discontinuities, together with the dominated convergence theorem. □

For $f \in L^2_\pm(\mathbb{R}^+)$, we define the smeared field operator

$$\tilde{A}(f,\varepsilon) = \int_0^\infty f(t)^*\tilde{a}_t(\varepsilon)\,dt. \tag{14}$$

It is straightforward to obtain the commutation relation

$$[\tilde{A}(f,\varepsilon), \tilde{A}(g,\varepsilon)^\dagger] = \int_0^\infty f(t)^* G_\varepsilon g(t)\, dt = \frac{1}{2}\int_0^\infty f(t)^*(g^+(t,\varepsilon) + g^-(t,\varepsilon))\, dt,$$

where we have written $G_\varepsilon g(t) = \int_0^\infty G_\varepsilon(t-s)g(s)\, ds = \frac{1}{2}g^+(t,\varepsilon) + \frac{1}{2}g^-(t,\varepsilon)$. Using (12), we may also express the smeared field as

$$\tilde{A}(f,\varepsilon) = A(f_\varepsilon^+) - \sqrt{\frac{\varepsilon}{\gamma}}\, f^+(0,\varepsilon)^* b, \tag{15}$$

where $f_\varepsilon^+(t) = f^+(t,\varepsilon)$. The second term ought to be negligible in the $\varepsilon \to 0^+$ limit and indeed, if $\varphi \in \mathfrak{h}_{\rm osc}$ is a vector with $\|b\varphi\| < \infty$ and if $e(g) = |g\rangle_{\rm resv}$ is an exponential vector ($g \in L^2(\mathbb{R}_+)$), then

$$\frac{\|(\tilde{A}(f,\varepsilon) - A(f))\,\varphi \otimes e(g)\|}{\|\varphi \otimes e(g)\|} \leq \frac{\|A(f_\varepsilon^+ - f)\, e(g)\|}{\|e(g)\|} + \sqrt{\frac{\varepsilon}{\gamma}}\, f^+(0,\varepsilon)^* \frac{\|b\varphi\|}{\|\varphi\|}$$
$$= \left|\int_0^\infty (f^+(t,\varepsilon) - f(t))^* g(t)\, dt\right| + \sqrt{\frac{\varepsilon}{\gamma}}\, f^+(0,\varepsilon)^* \frac{\|b\varphi\|}{\|\varphi\|} \xrightarrow{\varepsilon \to 0^+} 0.$$

More generally, we can define the smeared Weyl operators

$$\tilde{W}(f,\varepsilon) = \exp\{\tilde{A}(f,\varepsilon)^\dagger - \tilde{A}(f,\varepsilon)\}, \tag{16}$$

which satisfy the smeared canonical commutation relations

$$\tilde{W}(f,\varepsilon)\tilde{W}(g,\varepsilon) = \tilde{W}(f+g,\varepsilon)\, \exp\left\{-i\ \mathrm{Im}\int_0^\infty f(t)^* G_\varepsilon g(t)\, dt\right\}. \tag{17}$$

The smeared Weyl operator $\tilde{W}(f,\varepsilon)$ ought to converge to the standard Weyl unitary $W(f) = I_{\rm osc} \otimes \exp\{A(f)^\dagger - A(f)\}$ as $\varepsilon \to 0^+$. It is indeed not difficult to establish that for arbitrary $f_1, \cdots f_n \in L^2_\pm(\mathbb{R}_+)$

$$\|(\tilde{W}(f_1,\varepsilon)\cdots\tilde{W}(f_n,\varepsilon) - W(f_1)\cdots W(f_n))\,\varphi \otimes e(g)\| \xrightarrow{\varepsilon \to 0^+} 0.$$

This time, no restriction needs to be placed on $\varphi$. This is a type of quantum central limit theorem, however, it is less abstract than the "limit in matrix elements" traditionally encountered in the quantum probability literature[1,2,14] since the limit is taken on the fixed domain $\mathfrak{h}_{\rm osc}\underline{\otimes}\mathcal{E}_{\rm resv}$. The limiting operator is thus defined on the same Hilbert space, though it acts non-trivially on the noise space only.

## 4. LIMIT DYNAMICS

The limit of the process $\{\tilde{U}_t(\varepsilon) : t \geq 0\}$ as $\varepsilon \to 0^+$ is reminiscent of the Markov limits that have been widely studied in mathematical physics.[1,2,14] Comparison with previous results suggests that the limit be again described by a quantum stochastic process $\{\tilde{U}_t : t \geq 0\}$. We wish to deduce this limiting process by studying the limit of matrix elements $\langle \psi_1 | \tilde{U}_t(\varepsilon) \psi_2 \rangle$ for arbitrary vectors of the form

$$\psi_i = v_i \otimes |\alpha_i\rangle_{\text{osc}} \otimes |f_i\rangle_{\text{resv}}, \quad v_i \in \mathfrak{h}_{\text{sys}},\ \alpha_i \in \mathrm{C},\ f_i \in L^2_{\pm}(\mathbb{R}_+).$$

In other words, we would like to obtain $\tilde{U}_t$ as the weak limit of $\tilde{U}_t(\varepsilon)$, as $\varepsilon \to 0^+$, on the domain $\mathfrak{h}_{\text{sys}} \underline{\otimes} \mathcal{E}_{\text{osc}} \underline{\otimes} \mathcal{E}^{\pm}_{\text{resv}}$. We will similarly study weak limits of observables $\tilde{U}_t(\varepsilon)^\dagger E \tilde{U}_t(\varepsilon)$, where $E$ is a system observable, on the same domain.

Formally, we may expand $\tilde{U}_t(\varepsilon)$ as a Dyson series (by Picard iteration):

$$\tilde{U}_t(\varepsilon) = I + \sum_{n=1}^{\infty} (-i)^n \int_{\Delta_n(t)} ds_n \cdots ds_1\ \tilde{\Upsilon}_{s_n}(\varepsilon) \cdots \tilde{\Upsilon}_{s_1}(\varepsilon), \tag{18}$$

where the multi-time integrals are taken over the simplex

$$\Delta_n(t) = \{(s_n, \ldots, s_1) : t > s_n > \cdots > s_1 > 0\}.$$

The Dyson series expansion of $\langle \psi_1 | \tilde{U}_t(\varepsilon) \psi_2 \rangle$ is given by

$$\langle \psi_1 | \psi_2 \rangle + \sum_{n=1}^{\infty} (-i)^n \int_{\Delta_n(t)} ds_n \cdots ds_1\ \langle \psi_1 | \tilde{\Upsilon}_{s_n}(\varepsilon) \cdots \tilde{\Upsilon}_{s_1}(\varepsilon) \psi_2 \rangle. \tag{19}$$

The usual existence proof for differential equations by Picard iteration suggests that the Dyson series are convergent. Following Refs. 1, 2, 14, our basic approach will be as follows. First, we obtain an estimate of the form

$$\left| \int_{\Delta_n(t)} ds_n \cdots ds_1\ \langle \psi_1 | \tilde{\Upsilon}_{s_n}(\varepsilon) \cdots \tilde{\Upsilon}_{s_1}(\varepsilon) \psi_2 \rangle \right| \leq \Omega(n),$$

where $\Omega(n)$ is independent of $\varepsilon$ and such that $\sum_{n=1}^{\infty} \Omega(n) < \infty$. This establishes uniform convergence of the Dyson series (19) on $\varepsilon > 0$ (by the Weierstrass M-test). Consequently, we may exchange the limit and the summation in the Dyson series: i.e., we have established that $\langle \psi_1 | \tilde{U}_t(\varepsilon) \psi_2 \rangle$ converges as $\varepsilon \to 0^+$ to

$$\langle \psi_1 | \psi_2 \rangle + \sum_{n=1}^{\infty} (-i)^n \lim_{\varepsilon \to 0^+} \int_{\Delta_n(t)} ds_n \cdots ds_1\ \langle \psi_1 | \tilde{\Upsilon}_{s_n}(\varepsilon) \cdots \tilde{\Upsilon}_{s_1}(\varepsilon) \psi_2 \rangle.$$

It then remains to determine the limiting form of every term in the Dyson series individually. Summing these we obtain an (absolutely convergent) series expansion for the limiting matrix element, which we identify as the Dyson series expansion of the solution $\tilde{U}_t$ of a particular quantum stochastic differential equation. This completes the proof. Details can be found in Sec. 6.

In principle we should establish the results sketched above for every pair of vectors $\psi_i$. It is convenient, however, to reduce the problem to the study of the vacuum matrix element only. To this end we use the following identity:

$$\psi_i = \exp\{\alpha_i b^\dagger\} \exp\{A(f_i)^\dagger\}\, v_i \otimes |0\rangle_{\rm osc} \otimes |0\rangle_{\rm resv}.$$

By commuting the operators $\exp\{\alpha_i b^\dagger\}$ and $\exp\{A(f_i)^\dagger\}$ past $\tilde{\Upsilon}_{s_n}(\varepsilon)\cdots\tilde{\Upsilon}_{s_1}(\varepsilon)$ we can express the matrix element in the integrand in terms of the vacuum, provided we make some simple modifications to $\tilde{\Upsilon}_t(\varepsilon)$. Hence we have to go through the proofs only once using the vectors $v_i \otimes |0\rangle_{\rm osc} \otimes |0\rangle_{\rm resv}$.

**Lemma 3.** *Define* $\Phi = |0\rangle_{\rm osc} \otimes |0\rangle_{\rm resv}$. *Then*

$$\langle \psi_1 | \tilde{U}_t(\varepsilon)\psi_2\rangle = \langle v_1 \otimes \Phi | \check{U}_t(\varepsilon)\, v_2 \otimes \Phi\rangle\, \langle \alpha_1|\alpha_2\rangle_{\rm osc}\, \langle f_1|f_2\rangle_{\rm resv},$$

*where* $\check{U}_t(\varepsilon)$ *is the modification of* $\tilde{U}_t(\varepsilon)$ *obtained by the replacements*

$$\tilde{a}_t(\varepsilon) \longmapsto \check{a}_t^-(\varepsilon) = \tilde{a}_t(\varepsilon) + f_2^-(t,\varepsilon) - \sqrt{\frac{4\varepsilon}{\gamma}}\, G_\varepsilon(t)\,\alpha_2,$$

$$\tilde{a}_t(\varepsilon)^\dagger \longmapsto \check{a}_t^+(\varepsilon) = \tilde{a}_t(\varepsilon)^\dagger + f_1^-(t,\varepsilon)^* - \sqrt{\frac{4\varepsilon}{\gamma}}\, G_\varepsilon(t)\,\alpha_1^*.$$

**Proof:** Key here are the simple identities (on $\mathcal{D}'$)

$$b e^{\alpha b^\dagger} = e^{\alpha b^\dagger}(b+\alpha), \quad A(g)e^{A(f)^\dagger} = e^{A(f)^\dagger}\left(A(g) + \int_0^\infty g(s)^* f(s)\,ds\right),$$

which allow us to write using (12)

$$\tilde{a}_t(\varepsilon) e^{\alpha b^\dagger} e^{A(f)^\dagger} = e^{\alpha b^\dagger} e^{A(f)^\dagger}\left(\tilde{a}_t(\varepsilon) + f^-(t,\varepsilon) - \sqrt{\frac{4\varepsilon}{\gamma}}\, G_\varepsilon(t)\,\alpha\right).$$

Hence starting from any term in the Dyson series of the form

$$\langle \psi_1 | \tilde{\Upsilon}_{s_n}(\varepsilon)\cdots\tilde{\Upsilon}_{s_1}(\varepsilon)\psi_2\rangle = \langle v_1 \otimes \Phi | e^{\alpha_1^* b} e^{A(f_1)} \tilde{\Upsilon}_{s_n}(\varepsilon)\cdots\tilde{\Upsilon}_{s_1}(\varepsilon) \times e^{A(f_2)^\dagger} e^{\alpha_2 b^\dagger}\, v_2 \otimes \Phi\rangle,$$

the result follows using the above relations if we use additionally that

$$e^{\alpha_1^* b} e^{A(f_1)} e^{A(f_2)^\dagger} e^{\alpha_2 b^\dagger} = e^{A(f_2)^\dagger} e^{\alpha_2 b^\dagger} e^{\alpha_1^* b} e^{A(f_1)}\, \langle \alpha_1|\alpha_2\rangle_{\rm osc}\, \langle f_1|f_2\rangle_{\rm resv}$$

and that $e^{\alpha_i^* b} e^{A(f_i)}\,\Phi = \Phi$. □

The Dyson expansion for the matrix element $\langle \psi_1 | \tilde{U}_t(\varepsilon) \psi_2 \rangle$ may now be written, up to a constant prefactor of $\langle \alpha_1 | \alpha_2 \rangle_{\text{osc}} \, \langle f_1 | f_2 \rangle_{\text{resv}}$, as

$$\langle v_1 | v_2 \rangle + \sum_{n=1}^{\infty} (-i)^n \int_{\Delta_n(t)} ds_n \cdots ds_1 \, \langle v_1 \otimes \Phi | \check{\Upsilon}_{s_n}(\varepsilon) \cdots \check{\Upsilon}_{s_1}(\varepsilon) \, v_2 \otimes \Phi \rangle.$$

Here $\check{\Upsilon}_t(\varepsilon)$ is obtained from $\tilde{\Upsilon}_t(\varepsilon)$ by making the translations above, that is,

$$\check{\Upsilon}_t(\varepsilon) = \frac{4}{\gamma} \, E_{11} \check{a}_t^+(\varepsilon) \check{a}_t^-(\varepsilon) - \frac{2}{\sqrt{\gamma}} \, E_{10} \check{a}_t^+(\varepsilon) - \frac{2}{\sqrt{\gamma}} \, E_{01} \check{a}_t^-(\varepsilon) + E_{00}. \tag{20}$$

As the new processes $\check{a}_t^{\pm}(\varepsilon)$ are linear in the original O–U processes, we may write

$$\check{\Upsilon}_t(\varepsilon) = \check{E}_{11}(t, \varepsilon) \tilde{a}_t(\varepsilon)^\dagger \tilde{a}_t(\varepsilon) + \check{E}_{10}(t, \varepsilon) \tilde{a}_t(\varepsilon)^\dagger + \check{E}_{01}(t, \varepsilon) \tilde{a}_t(\varepsilon) + \check{E}_{00}(t, \varepsilon).$$

The coefficients $\check{E}_{ij}(t, \varepsilon)$ are easily worked out, however, our main interest will be in their limit values: we have for (Lebesgue-)a.e. $t \in \mathbb{R}_+$

$$\check{E}_{11}(t, \varepsilon) = \check{E}_{11}(t) = \frac{4}{\gamma} \, E_{11},$$

$$\lim_{\varepsilon \to 0^+} \check{E}_{10}(t, \varepsilon) = \check{E}_{10}(t) = -\frac{2}{\sqrt{\gamma}} \, E_{10} + \frac{4}{\gamma} \, E_{11} f_2(t),$$

$$\lim_{\varepsilon \to 0^+} \check{E}_{01}(t, \varepsilon) = \check{E}_{01}(t) = -\frac{2}{\sqrt{\gamma}} \, E_{01} + \frac{4}{\gamma} \, f_1(t)^* E_{11},$$

$$\lim_{\varepsilon \to 0^+} \check{E}_{00}(t, \varepsilon) = \check{E}_{00}(t) = E_{00} - \frac{2}{\sqrt{\gamma}} \, E_{01} f_2(t) - \frac{2}{\sqrt{\gamma}} \, f_1(t)^* E_{10} + \frac{4}{\gamma} \, f_1(t)^* E_{11} f_2(t),$$

the limits being uniform in the strong topology. Note that these limits depend only on the functions $f_i$ describing the resevoir: the parameters $\alpha_i$ for the oscillator have disappeared, indicating that the oscillator is indeed eliminated as $\varepsilon \to 0^+$.

The expansion of $\langle \psi_1 | \tilde{U}_t(\varepsilon) \psi_2 \rangle / \langle \alpha_1 | \alpha_2 \rangle_{\text{osc}} \, \langle f_1 | f_2 \rangle_{\text{resv}}$ may now be written as

$$\begin{aligned} \langle v_1 | v_2 \rangle + \; & \sum_{n=1}^{\infty} (-i)^n \int_{\Delta_n(t)} ds_n \cdots ds_1 \\ & \times \sum_{\alpha_n \beta_n} \cdots \sum_{\alpha_1 \beta_1} \langle v_1 | \check{E}_{\alpha_n \beta_n}(s_n, \varepsilon) \cdots \check{E}_{\alpha_1 \beta_1}(s_1, \varepsilon) \, v_2 \rangle \\ & \times \langle \Phi | [\tilde{a}_{s_n}(\varepsilon)^\dagger]^{\alpha_n} [\tilde{a}_{s_n}(\varepsilon)]^{\beta_n} \cdots [\tilde{a}_{s_1}(\varepsilon)^\dagger]^{\alpha_1} [\tilde{a}_{s_1}(\varepsilon)]^{\beta_1} \, \Phi \rangle, \end{aligned}$$

where $\alpha_k$, $\beta_k$ are summed over the values 0, 1 and we write $[x]^0 = 1$, $[x]^1 = x$. It is this form of the expansion that will be most useful in the proofs (Sec. 6).

We can now state the main results.

**Theorem 1.** *Suppose the system operators $E_{ij}$ are bounded with $\|E_{11}\| < \gamma/2$. Then there exists a unitary quantum stochastic process $\{\tilde{U}_t : t \geq 0\}$ such that*

$$\lim_{\varepsilon \to 0^+} \langle \psi_1 | \tilde{U}_t(\varepsilon) \psi_2 \rangle = \langle \psi_1 | \tilde{U}_t \psi_2 \rangle$$

*for any pair of vectors $\psi_{1,2} \in \mathfrak{h}_{\rm sys} \underline{\otimes} \mathcal{E}_{\rm osc} \underline{\otimes} \mathcal{E}^{\pm}_{\rm resv}$. The process $\{\tilde{U}_t : t \geq 0\}$ satisfies a QSDE of Hudson–Parthasarathy type*

$$d\tilde{U}_t = \left\{ (\tilde{W} - I)\, d\Lambda_t + \tilde{L}\, dA_t^\dagger - \tilde{L}^\dagger \tilde{W}\, dA_t - \frac{1}{2} \tilde{L}^\dagger \tilde{L}\, dt - i\tilde{H}\, dt \right\} \tilde{U}_t, \quad \tilde{U}_0 = I,$$

*where the coefficients are given by the expressions*

$$\tilde{W} = \frac{\gamma/2 - iE_{11}}{\gamma/2 + iE_{11}}, \quad \tilde{L} = \frac{i\sqrt{\gamma}}{\gamma/2 + iE_{11}} E_{10},$$
$$\tilde{H} = E_{00} + \mathrm{Im}\left\{ E_{01}(\gamma/2 + iE_{11})^{-1} E_{10} \right\}.$$

**Theorem 2.** *Under the conditions of Thm. 1, we have convergence of the Heisenberg evolution: for every bounded operator $E$ on $\mathfrak{h}_{\rm sys}$ and $\psi_{1,2} \in \mathfrak{h}_{\rm sys} \underline{\otimes} \mathcal{E}_{\rm osc} \underline{\otimes} \mathcal{E}^{\pm}_{\rm resv}$*

$$\lim_{\varepsilon \to 0^+} \langle \psi_1 | U_t(\varepsilon)^\dagger E U_t(\varepsilon) \psi_2 \rangle = \lim_{\varepsilon \to 0^+} \langle \psi_1 | \tilde{U}_t(\varepsilon)^\dagger E \tilde{U}_t(\varepsilon) \psi_2 \rangle = \langle \psi_1 | \tilde{U}_t^\dagger E \tilde{U}_t \psi_2 \rangle.$$

Let us demonstrate these results for some models used in the physics literature.

*Example 1.* Doherty *et al.*[(7)] consider the following system, in our notation:

$$\gamma = 2\kappa, \quad H = E_{00} - \frac{g_0^2}{\Delta} \cos^2(k_L x)\, b^\dagger b,$$

where $x$ is the atomic position operator on $\mathfrak{h}_{\rm sys} = L^2(\mathbb{R})$ and $E_{00}$ is a free Hamiltonian.[7] According to Theorem 1, the limiting time evolution is given by

$$d\tilde{U}_t = \left\{ (\tilde{W} - I)\, d\Lambda_t - iE_{00}\, dt \right\} \tilde{U}_t,$$
$$\tilde{W} = \frac{\kappa + ig_0^2 \cos^2(k_L x)/\Delta}{\kappa - ig_0^2 \cos^2(k_L x)/\Delta},$$

[7] Technically their Hamiltonian $E_{00} = p_x^2/2m$ is unbounded, but we sweep this under the rug.

provided that $\|E_{11}\| = g_0^2/\Delta < \kappa$. According to Theorem 2 and the quantum Itô rules, the limiting Heisenberg evolution of an atomic operator $X$ is given by

$$d(\tilde{U}_t^\dagger X \tilde{U}_t) = \tilde{U}_t^\dagger \left( i[E_{00}, X]\, dt + (\tilde{W}^\dagger X \tilde{W} - X)\, d\Lambda_t \right) \tilde{U}_t.$$

If we formally set $X$ to be the atomic momentum operator, then this expression is precisely Eq. (2.16ab) in Ref. 7 (taking into account the identity $\exp\{2i \tan^{-1}(x)\} = (\cos\{\tan^{-1}(x)\} + i \sin\{\tan^{-1}(x)\})^2 = (1+ix)/(1-ix)$). □

*Example 2.* The following interaction Hamiltonian is often used to describe the coupling between a collection of atomic spins (total spin $J$, i.e. $\mathfrak{h}_{\text{sys}} = \mathbb{C}^{2J+1}$) and a far detuned driven cavity mode (see e.g. Ref. 22):

$$H = \chi F_z b^\dagger b + \mathcal{U}(b^\dagger + b) + E_{00}.$$

Here $E_{00}$ is a free atomic Hamiltonian and $\chi, \mathcal{U}$ are real constants. By Theorem 1, the operators $\tilde{W}$, $\tilde{L}$, $\tilde{H}$ in the limiting QSDE become

$$\tilde{W} = \frac{\gamma/2 - i\chi F_z}{\gamma/2 + i\chi F_z}, \qquad \tilde{L} = \frac{i\mathcal{U}\sqrt{\gamma}}{\gamma/2 + i\chi F_z}, \qquad \tilde{H} = E_{00} - \frac{\chi \mathcal{U}^2 F_z}{\gamma^2/4 + \chi^2 F_z^2},$$

provided $\|E_{11}\| = \chi J < \gamma/2$. A common assumption in the literature (a reasonable one if the adiabatic approximation is good) is that $\|2\chi F_z/\gamma\| = 2\chi J/\gamma \ll 1$; the conventional adiabatically eliminated master equation (such as the one used in Ref. 22) is now recovered by calculating the master equation corresponding to $\tilde{U}_t$, then expanding $\tilde{L}$ and $\tilde{H}$ to first order with respect to $2\chi F_z/\gamma$. □

## 5. LIMIT OUTPUT FIELDS

Aside from the limit dynamics of the system observables, as in Theorem 2, we are also interested in the limiting behavior of the resevoir observables after interaction with the system and oscillator. In optical systems, for example, these observables can be detected (using, e.g., homodyne detection[3]) and the observed photocurrent can be used for statistical inference of the unmeasured system observables (quantum filtering theory[4,6]). The behavior of these observables in the singular limit is thus of significant interest for the modelling of quantum measurements.

To investigate the limit of the field observables we will study the convergence of matrix elements of the form $\langle \psi_1 | U_t(\varepsilon)^\dagger W(g_{t]}) U_t(\varepsilon)\, \psi_2 \rangle$, where $\psi_{1,2} \in \mathfrak{h}_{\text{sys}} \underline{\otimes} \mathcal{E}_{\text{osc}} \underline{\otimes} \mathcal{E}_{\text{resv}}^{\pm}$, $W(g)$ is the usual Weyl operator with $g \in L^2_{\pm}(\mathbb{R}_+)$ and $g_{t]}(s) = g(s)\chi_{[0,t]}(s)$. Note that unlike in the system operator case, $V_t(\varepsilon)$ does not commute with $W(g_{t]})$. However, we obtain using the quantum Itô rules

$$V_t(\varepsilon)^\dagger W(g_{t]}) V_t(\varepsilon) = \exp\left\{ A(g_{t]})^\dagger - A(g_{t]}) - 2(\tilde{A}(g_{t]}, \varepsilon)^\dagger - \tilde{A}(g_{t]}, \varepsilon)) \right\},$$

or, expressing this in terms of the usual Weyl operators,

$$V_t(\varepsilon)^\dagger W(g_{t]})V_t(\varepsilon) = W(g_{t]} - 2g^+_{t],\varepsilon}) \exp\left\{\sqrt{\frac{4\varepsilon}{\gamma}}\,(g^+_{t]}(0,\varepsilon)b^\dagger - g^+_{t]}(0,\varepsilon)^* b)\right\}.$$

Hence we can write

$$U_t(\varepsilon)^\dagger W(g_{t]})U_t(\varepsilon) = \tilde{U}_t(\varepsilon)^\dagger W(g_{t]} - 2g^+_{t],\varepsilon})B\big(\sqrt{4\varepsilon/\gamma}\,g^+_{t]}(0,\varepsilon)\big)\tilde{U}_t(\varepsilon), \tag{21}$$

where denote by $B(\alpha) = \exp\{\alpha b^\dagger - \alpha^* b\}$ the Weyl operator for the oscillator. Using the Dyson series for $\tilde{U}_t(\varepsilon)$, we now expand $\langle\psi_1|U_t(\varepsilon)^\dagger W(g_{t]})U_t(\varepsilon)\psi_2\rangle$ as

$$\sum_{n=0}^\infty\sum_{m=0}^\infty (-i)^{n-m}\int_{\Delta_m(t)} dt_m\cdots dt_1\int_{\Delta_n(t)} ds_n\cdots ds_1$$
$$\times\langle\psi_1|\tilde{\Upsilon}_{t_1}(\varepsilon)\cdots\tilde{\Upsilon}_{t_m}(\varepsilon)W(g_{t]} - 2g^+_{t],\varepsilon})B\big(\sqrt{4\varepsilon/\gamma}\,g^+_{t]}(0,\varepsilon)\big)\tilde{\Upsilon}_{s_n}(\varepsilon)\cdots\tilde{\Upsilon}_{s_1}(\varepsilon)\psi_2\rangle$$

(for notational simplicity we have used the convention $\int_{\Delta_0(t)}\cdots = I$ here). The limit of this expression is most easily studied by commuting the Weyl operators through the $\tilde{\Upsilon}_s(\varepsilon)$ terms, in the spirit of Lemma 3. In particular, using

$$W(f) = \exp\{A(f)^\dagger\}\exp\{-A(f)\}\exp\left\{-\frac{1}{2}\int_0^\infty |f(t)|^2\,dt\right\},$$

and similarly

$$B(\alpha) = \exp\{\alpha b^\dagger\}\exp\{-\alpha^* b\}\;\exp\{-|\alpha|^2/2\},$$

then moving the conjugated terms to the left and the remaining terms to the right (where they operate trivially on the vacuum), the problem can be reduced to the manipulations used in the proof of Theorem 2. Details can be found in Sec. 6.

We can already guess at this point, however, what the answer should be. As $g_{t]}(s) - 2g^+_{t]}(s,\varepsilon) \to -g_{t]}(s)$ $s$-a.e., and as $\sqrt{4\varepsilon/\gamma}\,g^+_{t]}(0,\varepsilon) \to 0$, we expect that

$$\tilde{U}_t(\varepsilon)^\dagger W(g_{t]} - 2g^+_{t],\varepsilon})B(\sqrt{4\varepsilon/\gamma}\,g^+_{t]}(0,\varepsilon))\tilde{U}_t(\varepsilon) \xrightarrow{\varepsilon\to 0^+} \tilde{U}_t^\dagger W(-g_{t]})\tilde{U}_t.$$

This is in fact the case.

**Theorem 3.** *Under the conditions of Theorem 1, we have the following: for any* $\psi_{1,2} \in \mathfrak{h}_{\mathrm{sys}}\underline{\otimes}\mathcal{E}_{\mathrm{osc}}\underline{\otimes}\mathcal{E}^\pm_{\mathrm{resv}}$ *and* $g \in L^2_\pm(\mathbb{R}_+)$

$$\lim_{\varepsilon\to 0^+}\langle\psi_1|U_t(\varepsilon)^\dagger W(g_{t]})U_t(\varepsilon)\psi_2\rangle = \langle\psi_1|\tilde{U}_t^\dagger W(-g_{t]})\tilde{U}_t\psi_2\rangle.$$

## 6. PROOFS

In the previous sections we have set up the problems to be solved, and we have investigated in detail the Ornstein–Uhlenbeck noises and the associated correlation functions and limits. With this preliminary spade work at hand, the remaining (technical) part of the proofs, as outlined in Sec. 4, follows to a large extent from the proofs and estimates in Ref. 14. Below we work through the required steps in the proofs, however we refer to Ref. 14 for some detailed calculations.

### 6.1. Proof of Theorem 1

#### *6.1.1. Wick Ordering*

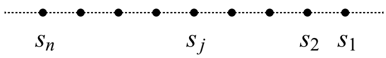
All steps of the proofs require us to evaluate the matrix elements

$$\langle\Phi|[\tilde{a}_{s_n}(\varepsilon)^\dagger]^{\alpha_n}[\tilde{a}_{s_n}(\varepsilon)]^{\beta_n}\cdots[\tilde{a}_{s_1}(\varepsilon)^\dagger]^{\alpha_1}[\tilde{a}_{s_1}(\varepsilon)]^{\beta_1}\,\Phi\rangle$$

that appear in the Dyson series. The solution to this problem is well known and proceeds by applying Wick's lemma.[(21)] For given sequences $\alpha = (\alpha_i)$, $\beta = (\beta_i)$, define the sets $P(\alpha) = \{i : \alpha_i = 1\}$ and $Q(\beta) = \{i : \beta_i = 1\}$. Let $\mathfrak{J}(\alpha, \beta)$ be the set of all maps $J : P(\alpha) \to Q(\beta)$ that are bijections and that are increasing, i.e. $J(i) > i$. Then by Wick's lemma we obtain

$$\langle\Phi|[\tilde{a}_{s_n}(\varepsilon)^\dagger]^{\alpha_n}[\tilde{a}_{s_n}(\varepsilon)]^{\beta_n}\cdots[\tilde{a}_{s_1}(\varepsilon)^\dagger]^{\alpha_1}[\tilde{a}_{s_1}(\varepsilon)]^{\beta_1}\,\Phi\rangle = \sum_{J\in\mathfrak{J}(\alpha,\beta)}\prod_{i\in P(\alpha)} G_\varepsilon(s_{J(i)} - s_i).$$

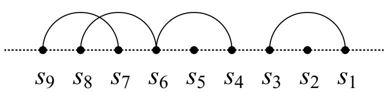
Diagrammatically, this can be represented as follows. Write $n$ vertices on a line:




For each vertex $j$, draw ingoing and outcoming lines corresponding to the values of $\alpha_j$ and $\beta_j$, as follows:

$\alpha_j = \beta_j = 1$ $\quad\alpha_j = 1, \beta_j = 0$ $\quad\alpha_j = 0, \beta_j = 1$ $\quad\alpha_j = \beta_j = 0$


Note that $P(\alpha)$ is the set of vertices that have outgoing lines, whereas $Q(\beta)$ is the set of vertices that have incoming lines. Next, connect every outgoing line to one of the incoming lines at a later time (i.e. form pair contractions), in such a way that all the lines are connected to exactly one other line. For example:



The ways in which this can be done are in one-to-one correspondence with the elements of $\mathfrak{J}(\alpha, \beta)$; the contracted vertices are then simply the pairs $(i, J(i))$ where $i \in P(\alpha)$. Wick's lemma tells us that the sum over all such (Goldstone) diagrams gives precisely the vacuum matrix element we are seeking.

### *6.1.2. Step 1: A Uniform Estimate*

Our first goal is to find a uniform (in $\varepsilon$) estimate on every term $M_\varepsilon(n)$ in the Dyson series:

$$M_\varepsilon(n) = \left| \int_{\Delta_n(t)} ds_n \cdots ds_1 \sum_{\alpha_n \beta_n} \cdots \sum_{\alpha_1 \beta_1} \langle v_1 | \check{E}_{\alpha_n \beta_n}(s_n, \varepsilon) \cdots \check{E}_{\alpha_1 \beta_1}(s_1, \varepsilon)\, v_2 \rangle \right.$$
$$\left. \times \langle \Phi | [\tilde{a}_{s_n}(\varepsilon)^\dagger]^{\alpha_n} [\tilde{a}_{s_n}(\varepsilon)]^{\beta_n} \cdots [\tilde{a}_{s_1}(\varepsilon)^\dagger]^{\alpha_1} [\tilde{a}_{s_1}(\varepsilon)]^{\beta_1}\, \Phi \rangle \right| .$$

Note that in this expression there is a summation over $\alpha$ and $\beta$. Hence every $n$-vertex Goldstone diagram is going to appear in the sum when we apply Wick's lemma, not just those with fixed incoming/outcoming lines for each vertex (as for fixed $\alpha$, $\beta$). Whenever this is the case it is convenient, rather than first summing over $\mathfrak{J}(\alpha, \beta)$ and then over $\alpha$, $\beta$, to arrange the sum in a slightly different way.

Every $n$-vertex Goldstone diagram can be described completely by specifying a partition of the set $\{1, \ldots, n\}$; each part of the partition corresponds to a group of vertices that are connected. For example, the nine-vertex example diagram above corresponds to the partition $\{\{1, 3\}, \{2\}, \{4, 6, 8\}, \{5\}, \{7, 9\}\}$. The corresponding values of $\alpha$ and $\beta$ are easily reconstructed: a singleton vertex has $\alpha = \beta = 0$, and for a doubleton or higher the first vertex has $\alpha = 1$, $\beta = 0$, the last vertex has $\alpha = 0$, $\beta = 1$, and the vertices in the middle have $\alpha = \beta = 1$. The sum over $\alpha$, $\beta$ and $\mathfrak{J}(\alpha, \beta)$, which appears in the expression for $M_\varepsilon(n)$ after applying Wick's lemma, can now be replaced by the sum over all partitions $\mathfrak{B}_n$ of the $n$-point set.

We need to refine the summation a little further. To every partition we associate a sequence $\mathbf{n} = (n_j)_{j \in \mathrm{N}}$ of integers, where $n_j$ counts the number of $j$-tuples that make up the partition (e.g., the example diagram above has $\mathbf{n} = (2, 2, 1, 0, 0, 0, \ldots)$). We denote by $E(\mathbf{n}) = \sum_j j n_j$ the number of vertices in the partition (so $E(\mathbf{n}) = n$ for a partition of $n$ vertices) and by $N(\mathbf{n}) = \sum_j n_j$ the number of parts that make up the partition. Of course there are many partitions that have the same occupation sequence $\mathbf{n}$; the set of all such partitions is denoted $\mathfrak{B}_{\mathbf{n}} \subset \mathfrak{B}_{E(\mathbf{n})}$. Summing over $\mathfrak{B}_n$ now corresponds to summing first over $\mathfrak{B}_{\mathbf{n}}$, then over all $\mathbf{n}$ with $E(\mathbf{n}) = n$.

We are now ready to bound $M_\varepsilon(n)$. First, note that

$$|\langle v_1 | \check{E}_{\alpha_n \beta_n}(s_n, \varepsilon) \cdots \check{E}_{\alpha_1 \beta_1}(s_1, \varepsilon)\, v_2 \rangle| \le \|v_1\|\, \|v_2\|\, C_{\alpha_n \beta_n} \cdots C_{\alpha_1 \beta_1},$$

where $C_{\alpha\beta}$ are finite positive constants that depend only on $\|E_{\alpha\beta}\|$, $\max_{[0,t]} f_{1,2}$ and $\gamma$. In particular, $C_{11} = (4/\gamma)\|E_{11}\|$ and we will write $C = \max_{\alpha\beta} C_{\alpha\beta}$. For any $\alpha, \beta$ corresponding to the occupation sequence $\mathbf{n}$, the number of times that $\alpha = 1, \beta = 1$ will be $\sum_{j>2}(j-2)n_j = E(\mathbf{n}) - N(\mathbf{n}) + n_1$. Hence

$$C_{\alpha_n\beta_n} \cdots C_{\alpha_1\beta_1} \leq C_{11}^{E(\mathbf{n})-N(\mathbf{n})+n_1} C^{N(\mathbf{n})-n_1}.$$

We can thus write

$$\frac{M_\varepsilon(n)}{\|v_1\|\,\|v_2\|} \leq \sum_{\mathbf{n}}^{E(\mathbf{n})=n} C_{11}^{E(\mathbf{n})-N(\mathbf{n})+n_1} C^{N(\mathbf{n})-n_1} \sum_{\rho\in\mathfrak{B}_\mathbf{n}} \times \int_{\Delta_n(t)} ds_n \cdots ds_1 \prod_{(i,j)\sim\rho} G_\varepsilon(s_i - s_j),$$

where $(i, j) \sim \rho$ denotes that the vertices $i$ and $j$ are contracted in the partition $\rho$. A clever argument due to Pulé can now be extended to show that

$$\sum_{\rho\in\mathfrak{B}_\mathbf{n}} \int_{\Delta_n(t)} ds_n \cdots ds_1 \prod_{(i,j)\sim\rho} G_\varepsilon(s_i - s_j) \leq \frac{1}{n_1!n_2!\cdots} \frac{t^{N(\mathbf{n})}}{2^{E(\mathbf{n})-N(\mathbf{n})}}.$$

Essentially, the trick is to rewrite the sum over $\mathfrak{B}_\mathbf{n}$ of integrals over the simplex as a single integral over a union of simplices, which can then be estimated; see Ref. 14, Sec. 7 for details. If $C_{11} > 0$, we obtain the following estimate uniformly in $\varepsilon$:

$$M_\varepsilon(n) \leq \Omega(n) = \|v_1\|\,\|v_2\| \sum_{\mathbf{n}}^{E(\mathbf{n})=n} \frac{e^{AE(\mathbf{n})+BN(\mathbf{n})}}{n_1!n_2!\cdots},$$

where $A = \log(C_{11}/2)$ and $B = \log(t \vee 1) + \log(C^2 \vee 1) + \log(C_{11}^{-2} \vee 1) + \log 2$. Summing over $n$, we obtain

$$\frac{1}{\|v_1\|\,\|v_2\|}\sum_n \Omega(n) = \sum_{\mathbf{n}} \frac{e^{AE(\mathbf{n})+BN(\mathbf{n})}}{n_1!n_2!\cdots} = \prod_{k=1}^{\infty}\sum_{n=0}^{\infty} \frac{e^{(kA+B)n}}{n!} = \exp\left\{\frac{e^{A+B}}{1-e^A}\right\},$$

provided that $e^A = C_{11}/2 < 1$, i.e. the sum converges provided that $\|E_{11}\| < \gamma/2$. Recall that this was a condition of Theorem 1. If $C_{11} = 0$ we obtain a slightly different estimate, which is however even simpler to sum (most terms vanish).

Now that we have a uniform estimate, the Weierstrass M-test guarantees that the Dyson series converges uniformly in $\varepsilon$. Consequently, we can calculate the limit of the Dyson series as $\varepsilon \to 0^+$ simply by calculating the limit of each diagram independently, then summing all these terms. This is what we will do below.

### *6.1.3. Step 2: Principal Terms in the Dyson Series*

The contribution of a single Goldstone diagram to the Dyson series has the form

$$\int_{\Delta_n(t)} ds_n \cdots ds_1 \langle v_1 | \check{E}_{\alpha_n \beta_n}(s_n, \varepsilon) \cdots \check{E}_{\alpha_1 \beta_1}(s_1, \varepsilon)\, v_2 \rangle \prod_{i \in P(\alpha)} G_\varepsilon(s_{J(i)} - s_i)$$

for some $J \in \mathfrak{J}(\alpha, \beta)$. A diagram will be called time-consecutive if $J(i) = i + 1$ for every $i \in P(\alpha)$. We claim that in the limit $\varepsilon \to 0^+$ any diagram that is not time-consecutive vanishes: hence we only need to retain time-consecutive diagrams.

To see this, first note that the magnitude of the diagram above is bounded by

$$C^n \, \|v_1\| \, \|v_2\| \int_{\Delta_n(t)} ds_n \cdots ds_1 \prod_{i \in P(\alpha)} G_\varepsilon\big(s_{J(i)} - s_i\big).$$

The limit of the latter integral is not difficult to evaluate explicitly. In particular, if $J$ is not time-consecutive then the integral vanishes in the limit $\varepsilon \to 0^+$. For example, suppose that $J(i) \neq i + 1$, so that $s_{J(i)} > s_{i+1}$ a.e. in $\Delta_n(t)$. Then

$$\int_0^{s_{i+1}} ds_i \; G_\varepsilon(s_{J(i)} - s_i) \xrightarrow{\varepsilon \to 0^+} 0 \quad \text{for any} \quad s_{J(i)} > s_{i+1}$$

by dominated convergence, as $G_\varepsilon(s_{J(i)} - s_i)$ is uniformly bounded on $[0, s_{i+1}]$ whenever $s_{J(i)} > s_{i+1}$ and $G_\varepsilon(s_{J(i)} - s_i) \to 0$ pointwise. On the other hand,

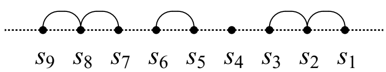
$$\int_0^{s_{i+1}} ds_i \; G_\varepsilon(s_{J(i)} - s_i) \leq \int_{-\infty}^{s_{J(i)}} ds_i \; G_\varepsilon(s_{J(i)} - s_i) = \frac{1}{2} \quad \text{for any} \quad s_{J(i)} s_{i+1}.$$

Hence we have by dominated convergence

$$\int_0^{s_{J(i)+1}} ds_{J(i)} \cdots \int_0^{s_{i+1}} ds_i \; G_\varepsilon\big(s_{J(i)} - s_i\big) \xrightarrow{\varepsilon \to 0^+} 0.$$

Proceeding in the same way, we can show that any diagram that is not time-consecutive vanishes as $\varepsilon \to 0^+$ (Ref. 14, Lemma 6.1).

It remains to consider the time-consecutive diagrams, for example:




These diagrams have a particularly simple structure: any such diagram is uniquely described by listing, in increasing time order, the number of vertices in each connected component. For example, the diagram above is described by the sequence (3, 1, 2, 3). In this way, any $n$-vertex diagram with $m$ connected components is described by a set of integers $r_1, \ldots, r_m$ such that $r_1 + \cdots + r_m = n$. Now suppose that $J \in \mathfrak{J}(\alpha, \beta)$ is a time-consecutive diagram that is described by the sequence

$r_1, \ldots, r_m$ with $r_1 + \cdots + r_m = n$. It is not difficult to verify that

$$\int_{\Delta_n(t)} ds_n \cdots ds_1 \; \Xi(s_1, \ldots, s_n) \prod_{i \in P(\alpha)} G_\varepsilon(s_{J(i)} - s_i) \xrightarrow{\varepsilon \to 0^+}$$

$$\frac{1}{2^{n-m}} \int_{\Delta_m(t)} dt_m \cdots dt_1 \; \Xi(\underbrace{t_1, t_1, \ldots, t_1}_{r_1 \text{ times}}, \underbrace{t_2, t_2, \ldots, t_2}_{r_2 \text{ times}}, \ldots, \underbrace{t_m, t_m, \ldots, t_m}_{r_m \text{ times}})$$

for any function $\Xi \in L^2_\pm(\mathbb{R}^n_+)$. Note that $n - m$ is precisely the number of contractions in the diagram $r_1, \ldots, r_m$.

### *6.1.4. Step 3: Resumming the Dyson Series*

We now compose the various steps made thus far. Starting from the $n$th term in the Dyson expansion, using Wick's lemma, retaining only the time-consecutive terms, and taking the limit as $\varepsilon \to 0^+$ gives

$$\int_{\Delta_n(t)} ds_n \cdots ds_1 \sum_{\alpha_n \beta_n} \cdots \sum_{\alpha_1 \beta_1} \langle v_1 | \check{E}_{\alpha_n \beta_n}(s_n, \varepsilon) \cdots \check{E}_{\alpha_1 \beta_1}(s_1, \varepsilon) \, v_2 \rangle$$
$$\times \langle \Phi | [\tilde{a}_{s_n}(\varepsilon)^\dagger]^{\alpha_n} [\tilde{a}_{s_n}(\varepsilon)]^{\beta_n} \cdots [\tilde{a}_{s_1}(\varepsilon)^\dagger]^{\alpha_1} [\tilde{a}_{s_1}(\varepsilon)]^{\beta_1} \, \Phi \rangle$$
$$\xrightarrow{\varepsilon \to 0^+} \sum_m \sum_{r_1, \ldots, r_m \geq 1}^{r_1 + \cdots + r_m = n} \frac{1}{2^{n-m}} \int_{\Delta_m(t)} dt_m \cdots dt_1 \left\langle v_1 | \check{E}^{(r_m)}(t_m) \cdots \check{E}^{(r_1)}(t_1) \, v_2 \right\rangle,$$

where we have written

$$\check{E}^{(r)}(t) = \begin{cases} \check{E}_{00}(t) & r = 1, \\ \check{E}_{01}(t)(\check{E}_{11}(t))^{r-2} \check{E}_{10}(t) & r \geq 2. \end{cases}$$

Let us now sum all the terms in the limiting Dyson series: this gives

$$\langle v_1, v_2 \rangle + \sum_{n=1}^{\infty} (-i)^n \sum_m \sum_{r_1, \ldots, r_m \geq 1}^{r_1 + \cdots + r_m = n} \frac{1}{2^{n-m}}$$
$$\times \int_{\Delta_m(t)} dt_m \cdots dt_1 \, \langle v_1 | \check{E}^{(r_m)}(t_m) \cdots \check{E}^{(r_1)}(t_1) \, v_2 \rangle.$$

Now use the fact that $n - m = \sum_k (r_k - 1)$ to rewrite this expression as

$$\langle v_1, v_2 \rangle + \sum_m \int_{\Delta_m(t)} dt_m \cdots dt_1 \left\langle v_1 | \left( \sum_{r_m \geq 1} \frac{\check{E}^{(r_m)}(t_m)}{i^{r_m} 2^{r_m - 1}} \right) \cdots \left( \sum_{r_1 \geq 1} \frac{\check{E}^{(r_1)}(t_1)}{i^{r_1} 2^{r_1 - 1}} \right) v_2 \right\rangle.$$

But note that we can sum

$$\sum_{r\geq 1} \frac{\check{E}^{(r)}(t)}{i^r 2^{r-1}} = -i\check{E}_{00}(t) - \frac{1}{2}\check{E}_{01}(t)\left(\sum_{r\geq 0}\frac{(\check{E}_{11}(t))^r}{(2i)^r}\right)\check{E}_{10}(t)$$

$$= -i\check{E}_{00}(t) - \frac{1}{2}\check{E}_{01}(t)\,\frac{1}{1+i\check{E}_{11}(t)/2}\,\check{E}_{10}(t)$$

provided that $\|i\check{E}_{11}(t)/2\| = (2/\gamma)\|E_{11}\| < 1$, which was already required for uniform convergence of the Dyson series. Finally we define

$$L_{\alpha\beta} = \left[-i\,E_{\alpha\beta} - E_{\alpha 1}\,\frac{1}{\gamma/2 + iE_{11}}\,E_{1\beta}\right]\left(-\frac{2}{\sqrt{\gamma}}\right)^{\alpha+\beta},$$

and note that we can write

$$-i\check{E}_{00}(t) - \frac{1}{2}\check{E}_{01}(t)\,\frac{1}{1+i\check{E}_{11}(t)/2}\,\check{E}_{10}(t) = \sum_{\alpha\beta}[f_1(t)^*]^{\alpha} L_{\alpha\beta}[f_2(t)]^{\beta}.$$

Hence the Dyson expansion for $\langle\psi_1|\tilde{U}_t(\varepsilon)\psi_2\rangle/\langle\alpha_1|\alpha_2\rangle_{\text{osc}}\,\langle f_1|f_2\rangle_{\text{resv}}$ may be written, in the limit $\varepsilon \to 0^+$, as

$$\langle v_1, v_2\rangle + \sum_m \sum_{\alpha_m\beta_m}\cdots\sum_{\alpha_1\beta_1}\langle v_1|L_{\alpha_m\beta_m}\cdots L_{\alpha_1\beta_1}\,v_2\rangle$$
$$\times \int_{\Delta_m(t)} dt_m\cdots dt_1\,[f_1(t_m)^*]^{\alpha_m}[f_2(t_m)]^{\beta_m}\cdots[f_1(t_1)^*]^{\alpha_1}[f_2(t_1)]^{\beta_1}.$$

### *6.1.5. Step 4: The Limit Unitary*

It remains to investigate the relation of the limiting Dyson series given above to the unitary evolution $\tilde{U}_t$. Consider a Hudson–Parthasarathy equation of the form

$$d\tilde{U}_t = \left\{L_{11}\,d\Lambda_t + L_{10}\,dA_t^{\dagger} + L_{01}\,dA_t + L_{00}\,dt\right\}\tilde{U}_t.$$

By Picard iteration, the solution $\tilde{U}_t$ can be developed into its chaos expansion

$$\tilde{U}_t = I + \sum_m\sum_{\alpha_m\beta_m}\cdots\sum_{\alpha_1\beta_1}\int_{\Delta_m(t)} L_{\alpha_m\beta_m}\cdots L_{\alpha_1\beta_1}\,d\Lambda_{t_m}^{\alpha_m\beta_m}\cdots d\Lambda_{t_1}^{\alpha_1\beta_1},$$

where we have used the Evans notation $\Lambda_t^{11} = \Lambda_t$, $\Lambda_t^{10} = A_t^{\dagger}$, $\Lambda_t^{01} = A_t$, $\Lambda_t^{00} = t$ (see e.g. Ref. 19, p. 151). Using the usual formula for the matrix elements of stochastic integrals, it is evident that $\langle\psi_1|\tilde{U}_t\,\psi_2\rangle$ coincides with the limiting Dyson series above. It remains to notice, as is verified through straightforward manipulations, that $L_{11} = \tilde{W} - I$, $L_{10} = \tilde{L}$, $L_{01} = -\tilde{L}^\dagger \tilde{W}$, and $L_{00} = -i\tilde{H} - \tilde{L}^\dagger \tilde{L}/2$. The proof of Theorem 1 is complete.

## 6.2. Proof of Theorem 2

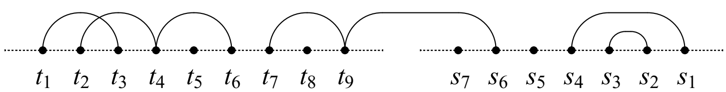
Conceptually, little changes when we are interested in the Heisenberg evolution. Using the Dyson series for $\tilde{U}_t(\varepsilon)$, we now expand $\langle \psi_1 | \tilde{U}_t(\varepsilon)^\dagger X \tilde{U}_t(\varepsilon) \psi_2 \rangle$ as

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-i)^{n-m} \int_{\Delta_m(t)} dt_m \cdots dt_1 \int_{\Delta_n(t)} ds_n \cdots ds_1$$
$$\times \langle \psi_1 | \tilde{\Upsilon}_{t_1}(\varepsilon) \cdots \tilde{\Upsilon}_{t_m}(\varepsilon) X \tilde{\Upsilon}_{s_n}(\varepsilon) \cdots \tilde{\Upsilon}_{s_1}(\varepsilon) \psi_2 \rangle$$

(for notational simplicity, we write from this point on $\int_{\Delta_0(t)} \cdots = I$). This equals

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (-i)^{n-m} \int_{\Delta_m(t)} dt_m \cdots dt_1 \int_{\Delta_n(t)} ds_n \cdots ds_1 \sum_{\mu_m \nu_m} \cdots \sum_{\mu_1 \nu_1} \sum_{\alpha_n \beta_n} \cdots \sum_{\alpha_1 \beta_1}$$
$$\times \langle \psi_1 | \check{E}_{\mu_1 \nu_1}(t_1, \varepsilon) \cdots \check{E}_{\mu_m \nu_m}(t_m, \varepsilon) X \check{E}_{\alpha_n \beta_n}(s_n, \varepsilon) \cdots \check{E}_{\alpha_1 \beta_1}(s_1, \varepsilon) \psi_2 \rangle$$
$$\times \langle \Phi | [\tilde{a}_{t_1}(\varepsilon)^\dagger]^{\mu_1} [\tilde{a}_{t_1}(\varepsilon)]^{\nu_1} \cdots [\tilde{a}_{t_m}(\varepsilon)^\dagger]^{\mu_m} [\tilde{a}_{t_m}(\varepsilon)]^{\nu_m}$$
$$\times [\tilde{a}_{s_n}(\varepsilon)^\dagger]^{\alpha_n} [\tilde{a}_{s_n}(\varepsilon)]^{\beta_n} \cdots [\tilde{a}_{s_1}(\varepsilon)^\dagger]^{\alpha_1} [\tilde{a}_{s_1}(\varepsilon)]^{\beta_1} \, \Phi \rangle,$$

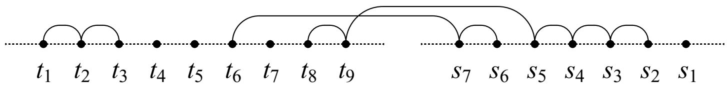
where we have applied Lemma 3. As before, we can use Wick's lemma to evaluate the vacuum matrix element. Drawing vertices on a line in the correct order, assigning incoming and outgoing lines according to $\alpha$, $\beta$, $\mu$, $\nu$, and connecting them up, allows us to represent the vacuum matrix element as a sum over the usual diagrams. For example, a possible diagram in this case might be:



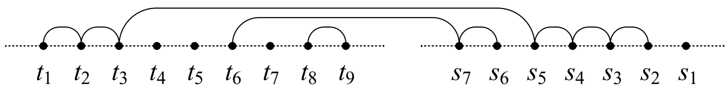

Note that we do not need to worry about the time ordering (which is obviously not satisfied in this case), as the commutators between $\tilde{a}_s(\varepsilon)$ and $\tilde{a}_t(\varepsilon)^\dagger$ are symmetric in $s, t$; hence only the order in which the $\tilde{a}$'s and $\tilde{a}^\dagger$'s occur will matter, and we can expand in terms of pair contractions in the usual way.

The first question that needs to be resolved is whether we still have uniform control on the convergence of the Dyson series. This does turn out to be the case. The argument used previously to obtain the required estimates can be generalized also to the Heisenberg evolution, though the details of the argument are somewhat more involved in this case. We refer to Ref. 14 for further details.

The next problem is to determine which diagrams survive in the $\varepsilon \to 0^+$ limit. It is not difficult to see that diagrams with contractions between $s$-variables which are not time-consecutive or between $t$-variables which are not time-consecutive will vanish in the limit; this follows directly from the previous arguments. Hence all surviving diagrams must have only time-consecutive contractions within the $s$- and $t$-blocks. On the other hand, note that we are not integrating over the simplex $\Delta_{m+n}(t)$, but rather over the product of simplices $\Delta_m(t) \times \Delta_n(t)$. Therefore contractions between $s$- and $t$-variables do not necessarily give vanishing contributions, provided that the corresponding lines in the diagram do not cross—in the latter case the contraction would force $s_i = t_l$ and $s_j = t_k$ in the limit $\varepsilon \to 0^+$, whereas integration over $\Delta_m(t) \times \Delta_n(t)$ requires $s_i < s_j$ and $t_k < t_l$. For example,




must necessarily vanish, whereas the diagram




could give a nonvanishing contribution to the Dyson expansion. To characterize such diagrams, we begin as before by specifying in increasing time order the numbers $r_1, \ldots, r_p$ of vertices connected through contractions within the $s$-block, and specifying the numbers $l_1, \ldots, l_q$ of vertices connected through contractions within the $t$-block, also in increasing time order. For example, the nonvanishing diagram above is described by the sequences $r = (1, 4, 2)$ and $l = (3, 1, 1, 1, 1, 2)$. This specifies completely the (time-consecutive) contractions within the $s$- and $t$-blocks.

It remains to specify the contractions between $s$- and $t$-variables. Note that we can only get additional contractions between the left endpoint of a connected component in the $s$-block with the right endpoint of a connected component in the $t$-block. Let us write $\kappa_i = 1$ if the (left endpoint of the) $i$th connected component in the $s$-block is contracted with a vertex in the $t$-block, and $\kappa_i = 0$ otherwise; similarly, we write $\lambda_i = 1$ if the (right endpoint of the) $i$th connected component in the $t$-block is contracted with a vertex in the $s$-block, and $\lambda_i = 0$ otherwise (note that necessarily $\sum \kappa = \sum \lambda$). For example, the nonvanishing diagram above is described by $\kappa = (0, 1, 1)$ and $\lambda = (1, 0, 0, 1, 0, 0)$. Finally, we denote by $\kappa(i)$ the $i$th nonzero element of $\kappa$, and similarly for $\lambda(i)$. For example, in the nonvanishing diagram above, $\kappa(1) = 2$, $\kappa(2) = 3$, and $\lambda(1) = 1$, $\lambda(2) = 4$. Once we have given $r$, $l$, $\kappa$ and $\lambda$ we have described uniquely one nonvanishing diagram, as the order in which the $s$–$t$ contractions are made is fixed by the requirement that the corresponding lines be noncrossing (we must connect the lines from the inside out, i.e. connected component $\kappa(i)$ is contracted with connected component $\lambda(i)$).

With this somewhat tedious notation, we can write out the limiting Dyson series explicitly. Applying Wick's lemma, retaining only the nonvanishing diagrams, and taking the limit as $\varepsilon \to 0^+$ gives

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\hat{n},\hat{m}}\sum_{r_1,\ldots,r_{\hat{n}}}^{\sum r=n}\sum_{l_1,\ldots,l_{\hat{m}}}^{\sum l=m}\frac{(-i)^{n-m}}{2^{n-\hat{n}}2^{m-\hat{m}}}\sum_{\kappa,\lambda}^{\sum\kappa=\sum\lambda}\int_{\Delta_{\hat{m}}(t)}dt_{\hat{m}}\cdots dt_1\int_{\Delta_{\hat{n}}(t)}ds_{\hat{n}}\cdots ds_1$$

$$\times\left\langle\psi_1|\check{E}^{(l_1)}_{0\lambda_1}(t_1)\cdots\check{E}^{(l_{\hat{m}})}_{0\lambda_{\hat{m}}}(t_{\hat{m}})X\check{E}^{(r_{\hat{n}})}_{\kappa_{\hat{n}}0}(s_{\hat{n}})\cdots\check{E}^{(r_1)}_{\kappa_1 0}(s_1)\psi_2\right\rangle\prod_i\delta(s_{\kappa(i)}-t_{\lambda(i)}),$$

where we have written

$$\check{E}^{(r)}_{\alpha\beta}(t)=\begin{cases}\check{E}_{\alpha\beta}(t) & r=1,\\ \check{E}_{\alpha 1}(t)(\check{E}_{11}(t))^{r-2}\check{E}_{1\beta}(t) & r\geq 2.\end{cases}$$

We could proceed at this point to resum the Dyson series as before, but instead it will be more convenient to work backwards from the desired result and show that we can recover the expression above.

Consider once more the Hudson–Parthasarathy equation

$$d\tilde{U}_t=\{L_{11}\,d\Lambda_t+L_{10}\,dA_t^\dagger+L_{01}\,dA_t+L_{00}\,dt\}\tilde{U}_t.$$

We are interested in the matrix element

$$\langle\psi_1|\tilde{U}_t^\dagger X\tilde{U}_t\,\psi_2\rangle=\langle\alpha_1|\alpha_2\rangle_{\rm osc}\big\langle v_1\otimes\Phi|e^{A(f_1)}\,\tilde{U}_t^\dagger X\tilde{U}_t\,e^{A(f_2)^\dagger}\,v_2\otimes\Phi\big\rangle.$$

Using the Itô rules, we can commute the field operators past the unitaries; then

$$\langle\psi_1|\tilde{U}_t^\dagger X\tilde{U}_t\,\psi_2\rangle=\langle\alpha_1|\alpha_2\rangle_{\rm osc}\langle f_1|f_2\rangle_{\rm resv}\langle v_1\otimes\Phi|\check{U}_t^+X\check{U}_t\,v_2\otimes\Phi\rangle,$$

where we have written

$$d\check{U}_t=\{\check{L}_{11}(t)\,d\Lambda_t+\check{L}_{10}(t)\,dA_t^\dagger+\check{L}_{01}(t)\,dA_t+\check{L}_{00}(t)\,dt\}\check{U}_t,$$
$$d\check{U}_t^+=\check{U}_t^+\{\check{L}_{11}^+(t)\,d\Lambda_t+\check{L}_{10}^+(t)\,dA_t^\dagger+\check{L}_{01}^+(t)\,dA_t+\check{L}_{00}^+(t)\,dt\},$$

and where the coefficients are given by

$$\check{L}_{11}(t)=L_{11},\quad \check{L}_{10}(t)=L_{10}+L_{11}f_2(t),\quad \check{L}_{01}(t)=L_{01}+f_1(t)^*L_{11},$$
$$\check{L}_{11}^+(t)=L_{11}^\dagger,\quad \check{L}_{10}^+(t)=L_{01}^\dagger+L_{11}^\dagger f_2(t),\quad \check{L}_{01}^+(t)=L_{10}^\dagger+f_1(t)^*L_{11}^\dagger,$$

and

$$\check{L}_{00}(t)=\sum_{\alpha\beta}[f_1(t)^*]^\alpha L_{\alpha\beta}[f_2(t)]^\beta,\quad \check{L}_{00}^+(t)=\sum_{\alpha\beta}[f_1(t)^*]^\alpha L_{\beta\alpha}^\dagger[f_2(t)]^\beta.$$

But by explicit summation one may verify that

$$\check{L}_{\alpha\beta}(t) = \sum_{r\geq 1} \frac{\check{E}^{(r)}_{\alpha\beta}(t)}{i^r 2^{r-1}}, \quad \check{L}^+_{\alpha\beta}(t) = \sum_{r\geq 1} \frac{\check{E}^{(r)}_{\alpha\beta}(t)}{(-i)^r 2^{r-1}}.$$

Using Picard iteration to develop $\check{U}_t$ and $\check{U}_t^+$ into their chaotic expansions, substituting the above expressions for $\check{L}$, $\check{L}^+$ and rearranging the summations somewhat, we arrive at the following Dyson expansion for $\langle v_1 \otimes \Phi | \check{U}_t^+ X \check{U}_t \, v_2 \otimes \Phi \rangle$:

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{\hat{n},\hat{m}} \sum_{r_1,\ldots,r_{\hat{n}}}^{\sum r=n} \sum_{l_1,\ldots,l_{\hat{m}}}^{\sum l=m} \frac{(-i)^{n-m}}{2^{n-\hat{n}} 2^{m-\hat{m}}} \sum_{\alpha_{\hat{n}}\beta_{\hat{n}}} \cdots \sum_{\alpha_1\beta_1} \sum_{\mu_{\hat{m}}\nu_{\hat{m}}} \cdots \sum_{\mu_1\nu_1}$$
$$\times \langle v_1 \otimes \Phi | \int_{\Delta_{\hat{m}}(t)} \check{E}^{(l_1)}_{\mu_1\nu_1}(t_1) \cdots \check{E}^{(l_{\hat{m}})}_{\mu_{\hat{m}}\nu_{\hat{m}}}(t_{\hat{m}}) \, d\Lambda^{\mu_{\hat{m}}\nu_{\hat{m}}}_{t_{\hat{m}}} \cdots d\Lambda^{\mu_1\nu_1}_{t_1}$$
$$\times X \times \int_{\Delta_{\hat{n}}(t)} \check{E}^{(r_{\hat{n}})}_{\alpha_{\hat{n}}\beta_{\hat{n}}}(s_{\hat{n}}) \cdots \check{E}^{(r_1)}_{\alpha_1\beta_1}(s_1) \, d\Lambda^{\alpha_{\hat{n}}\beta_{\hat{n}}}_{s_{\hat{n}}} \cdots d\Lambda^{\alpha_1\beta_1}_{s_1} \, v_2 \otimes \Phi \rangle.$$

Using the quantum Itô rules and by induction on the iterated integrals, it is not difficult to establish that the vacuum matrix element in this expression vanishes if any of the $\mu_i$ or $\beta_i$ are nonzero, or if the number of nonzero $\nu$'s and $\alpha$'s do not coincide. Hence we find, relabeling the variables suggestively,

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{\hat{n},\hat{m}} \sum_{r_1,\ldots,r_{\hat{n}}}^{\sum r=n} \sum_{l_1,\ldots,l_{\hat{m}}}^{\sum l=m} \frac{(-i)^{n-m}}{2^{n-\hat{n}} 2^{m-\hat{m}}} \sum_{\kappa,\lambda}^{\sum \kappa = \sum \lambda}$$
$$\times \langle v_1 \otimes \Phi | \int_{\Delta_{\hat{m}}(t)} \check{E}^{(l_1)}_{0\lambda_1}(t_1) \cdots \check{E}^{(l_{\hat{m}})}_{0\lambda_{\hat{m}}}(t_{\hat{m}}) \, d\Lambda^{0\lambda_{\hat{m}}}_{t_{\hat{m}}} \cdots d\Lambda^{0\lambda_1}_{t_1}$$
$$\times X \times \int_{\Delta_{\hat{n}}(t)} \check{E}^{(r_{\hat{n}})}_{\kappa_{\hat{n}}0}(s_{\hat{n}}) \cdots \check{E}^{(r_1)}_{\kappa_1 0}(s_1) \, d\Lambda^{\kappa_{\hat{n}}0}_{s_{\hat{n}}} \cdots d\Lambda^{\kappa_1 0}_{s_1} \, v_2 \otimes \Phi \rangle.$$

But now we can easily reduce to the previous form of the Dyson expansion, taking into account the identity (which follows directly from the quantum Itô rules)

$$\Big\langle v \otimes \Phi \Big| \int_0^t F_\tau \, dA_\tau \times \int_0^s G_\sigma \, dA^\dagger_\sigma \, w \otimes \Phi \Big\rangle$$
$$= \int_0^t d\tau \int_0^s d\sigma \, \langle v \otimes \Phi | F_\tau G_\sigma \, w \otimes \Phi \rangle \, \delta(\tau - \sigma).$$

The proof of Theorem 2 is complete.

## 6.3. Proof of Theorem 3

The hard work has already been done in the proof of Theorem 2; all we have to do to prove Theorem 3 is an appropriate shift of the coefficients. We briefly provide the details. Consider first the expansion for $\langle\psi_1|U_t(\varepsilon)^\dagger W(g_{t]})U_t(\varepsilon)\psi_2\rangle$,

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-i)^{n-m}\int_{\Delta_m(t)}dt_m\cdots dt_1\int_{\Delta_n(t)}ds_n\cdots ds_1$$
$$\times\langle v_1\otimes\Phi|\check{\Upsilon}_{t_1}(\varepsilon)\cdots\check{\Upsilon}_{t_m}(\varepsilon)W(g_{t]}-2g^+_{t],\varepsilon})B\left(\sqrt{\tfrac{4\varepsilon}{\gamma}}\,g^+_{t]}(0,\varepsilon)\right)\check{\Upsilon}_{s_n}(\varepsilon)\cdots\check{\Upsilon}_{s_1}(\varepsilon)v_2\otimes\Phi\rangle,$$

where we have dropped the prefactor $\langle\alpha_1|\alpha_2\rangle_{\text{osc}}\,\langle f_1|f_2\rangle_{\text{resv}}$ and the constant factor that is obtained from commuting $e^{A(f_1)}$, etc., past the Weyl operators. Splitting up the Weyl operators as explained in Sec. 5 and commuting them through the Hamiltonians $\check{\Upsilon}$ as in Lemma 3 gives

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-i)^{n-m}\int_{\Delta_m(t)}dt_m\cdots dt_1\int_{\Delta_n(t)}ds_n\cdots ds_1$$
$$\times\langle v_1\otimes\Phi|\check{\Upsilon}^\wedge_{t_1}(\varepsilon)\cdots\check{\Upsilon}^\wedge_{t_m}(\varepsilon)\check{\Upsilon}^\vee_{s_n}(\varepsilon)\cdots\check{\Upsilon}^\vee_{s_1}(\varepsilon)\,v_2\otimes\Phi\rangle,$$

where we have dropped the constant factor that is obtained when we split the Weyl operators. Here $\check{\Upsilon}^\wedge_s$ is obtained from $\check{\Upsilon}_s$ by transforming

$$\check{E}_{10}(s,\varepsilon)\mapsto\check{E}_{10}(s,\varepsilon)+\left[\{g_{t]}-2g^+_{t],\varepsilon}\}^-(s,\varepsilon)-\frac{4\varepsilon}{\gamma}\,g^+_{t]}(0,\varepsilon)\,G_\varepsilon(s)\right]\check{E}_{11}(s,\varepsilon),$$
$$\check{E}_{00}(s,\varepsilon)\mapsto\check{E}_{00}(s,\varepsilon)+\left[\{g_{t]}-2g^+_{t],\varepsilon}\}^-(s,\varepsilon)-\frac{4\varepsilon}{\gamma}\,g^+_{t]}(0,\varepsilon)\,G_\varepsilon(s)\right]\check{E}_{01}(s,\varepsilon),$$

and $\check{\Upsilon}^\vee_s$ is obtained from $\check{\Upsilon}_s$ by transforming

$$\check{E}_{01}(s,\varepsilon)\mapsto\check{E}_{01}(s,\varepsilon)-\left[\{g_{t]}-2g^+_{t],\varepsilon}\}^-(s,\varepsilon)^*-\frac{4\varepsilon}{\gamma}\,g^+_{t]}(0,\varepsilon)^*\,G_\varepsilon(s)\right]\check{E}_{11}(s,\varepsilon),$$
$$\check{E}_{00}(s,\varepsilon)\mapsto\check{E}_{00}(s,\varepsilon)-\left[\{g_{t]}-2g^+_{t],\varepsilon}\}^-(s,\varepsilon)^*-\frac{4\varepsilon}{\gamma}\,g^+_{t]}(0,\varepsilon)^*\,G_\varepsilon(s)\right]\check{E}_{10}(s,\varepsilon).$$

We can now proceed exactly as in the proof of Theorem 2 to establish that in the limit $\varepsilon\to 0^+$, this expansion reduces to

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{\hat{n},\hat{m}}\sum_{r_1,\ldots,r_{\hat{n}}}^{\sum r=n}\sum_{l_1,\ldots,l_{\hat{m}}}^{\sum l=m}\frac{(-i)^{n-m}}{2^{n-\hat{n}}2^{m-\hat{m}}}\sum_{\kappa,\lambda}^{\sum\kappa=\sum\lambda}\int_{\Delta_{\hat{m}}(t)}dt_{\hat{m}}\cdots dt_1\int_{\Delta_{\hat{n}}(t)}ds_{\hat{n}}\cdots ds_1$$
$$\times\left\langle v_1\left|\check{E}^{(l_1)\wedge}_{0\lambda_1}(t_1)\cdots\check{E}^{(l_{\hat{m}})\wedge}_{0\lambda_{\hat{m}}}(t_{\hat{m}})\check{E}^{(r_{\hat{n}})\vee}_{\kappa_{\hat{n}}0}(s_{\hat{n}})\cdots\check{E}^{(r_1)\vee}_{\kappa_1 0}(s_1)\,v_2\right.\right\rangle\prod_i\delta(s_{\kappa(i)}-t_{\lambda(i)}),$$

where $\check{E}^{(l)\wedge}_{0\lambda}(s)$ is obtained through the replacements

$$\check{E}_{10}(s) \mapsto \check{E}_{10}(s) - g(s)\check{E}_{11}(s), \quad \check{E}_{00}(s) \mapsto \check{E}_{00}(s) - g(s)\check{E}_{01}(s),$$

and $\check{E}^{(r)\vee}_{\kappa 0}(s)$ is obtained through the replacements

$$\check{E}_{01}(s) \mapsto \check{E}_{01}(s) + g(s)^*\check{E}_{11}(s), \quad \check{E}_{00}(s) \mapsto \check{E}_{00}(s) + g(s)^*\check{E}_{10}(s).$$

Starting from the opposite direction, it is not difficult to establish that the expected result of Theorem 3, $\langle \psi_1 | \tilde{U}_t^\dagger W(-g_{t]})\tilde{U}_t \psi_2\rangle$, can be written (modulo prefactor) as $\langle v_1 \otimes \Phi | \check{U}_t^\wedge \check{U}_t^\vee \, v_2 \otimes \Phi\rangle$, where $\check{U}_t^\wedge$ is obtained from $\check{U}_t^+$ by the replacements

$$\check{L}^+_{10}(s) \mapsto \check{L}^+_{10}(s) - g(s)\check{L}^+_{11}(s), \quad \check{L}^+_{00}(s) \mapsto \check{L}^+_{00}(s) - g(s)\check{L}^+_{01}(s),$$

and $\check{U}_t^\vee$ is obtained from $\check{U}_t$ by the replacements

$$\check{L}_{01}(s) \mapsto \check{L}_{01}(s) + g(s)^*\check{L}_{11}(s), \quad \check{L}_{00}(s) \mapsto \check{L}_{00}(s) + g(s)^*\check{L}_{10}(s).$$

It is important to note that the constant factor which we have dropped here is precisely the limit as $\varepsilon \to 0^+$ of the constant factor that was dropped previously; hence it suffices to show that the two expansions above coincide. However, this is immediate from our previous results, and the theorem is proved.

## APPENDIX A: SOME HEURISTIC CALCULATIONS

The goal of this appendix is to demonstrate our model and results through a formal computation, similar to the computations that might be found in papers on quantum optics. The aim is to help the reader who is unfamiliar with quantum stochastic calculus to make a connection with the physics literature. It should be emphasized, however, that everything we will do here is completely heuristic. In particular, the main part of this paper does not attempt to make rigorous the formal procedure used below, which is difficult to justify beyond heuristics; indeed, we will see that we can just as easily obtain an incorrect result through such a procedure. Instead, we pose the problem in a mathematically well defined manner in Sec. 2, which allows us to obtain rigorous limit results and provides additional physical insight into the nature of the problem.

In the physics literature, our basic model is often introduced formally as follows (see, e.g., Ref. 11, Sec. 5.3). Let $\mathsf{a}(\omega)$ be Boson annihilation operators with singular commutation relations $[\mathsf{a}(\omega), \mathsf{a}^\dagger(\omega')] = \delta(\omega - \omega')$, describing the Fourier modes of a quantum field. We could now describe the interaction of the atom-cavity system and the field by introducing the Hamiltonian $H_{\text{tot}} = H + H_{\text{field}} + H_{\text{int}}$, where the atom-cavity Hamiltonian $H$ is defined in Eq. (1), and the free field and cavity-field interaction Hamiltonians are given by

$$H_{\text{field}} = \int \omega\, \mathsf{a}^\dagger(\omega)\mathsf{a}(\omega)\, d\omega, \quad H_{\text{int}} = \int \sqrt{\frac{\gamma(\omega)}{2\pi}}\, \{ib\mathsf{a}^\dagger(\omega) - ib^\dagger \mathsf{a}(\omega)\}\, d\omega.$$

$H_{\text{int}}$ describes the coupling between the cavity and the field in the rotating wave approximation; $\gamma(\omega)$ is the coupling strength. Now let $U_t$ be the time evolution unitary obtained by solving the Schrödinger equation for this model in the *interaction picture* with respect to $H_{\text{field}}$. Evidently

$$\frac{d}{dt}U_t = \left[\int \sqrt{\frac{\gamma(\omega)}{2\pi}}\, \{b\mathsf{a}^\dagger(\omega)e^{i\omega t} - b^\dagger \mathsf{a}(\omega)e^{-i\omega t}\}\, d\omega - iH\right] U_t.$$

This model is not Markovian (i.e., if we were to trace over the bath, we would not obtain a Lindblad-type master equation), but the Markovian approximation corresponds to making $\gamma(\omega)$ independent of $\omega$. Adopting this approximation, we can write

$$\frac{d}{dt}U_t = \left[\sqrt{\gamma}\,\{b\mathsf{a}_t^\dagger - b^\dagger \mathsf{a}_t\} - iH\right]U_t, \quad \mathsf{a}_t = \frac{1}{\sqrt{2\pi}}\int \mathsf{a}(\omega)\, e^{-i\omega t}\, d\omega.$$

The process $\mathsf{a}_t$ satisfies the commutation relation $[\mathsf{a}_t, \mathsf{a}_s^\dagger] = \delta(t-s)$, so that it can be interpreted as a (Bosonic) quantum white noise. The equation for $U_t$ is now ambiguous, however; as in classical stochastic integration, inequivalent interpretations are possible (Itô vs. Stratonovich).

A useful heuristic for manipulations with quantum white noises is described in Refs. 13, 15. The upshot is that we should interpret time ordered equations, such as the previous equation for $U_t$, as Stratonovich equations. Itô equations, on the other hand, correspond to normal ordered equations. To write the equation for $U_t$ in normal ordered form, let us calculate $[\mathsf{a}_t, U_t]$:

$$[\mathsf{a}_t, U_t] = \int_0^t \left[\mathsf{a}_t, \sqrt{\gamma}\,\{b\mathsf{a}_s^\dagger - b^\dagger \mathsf{a}_s\}U_s - iHU_s\right] ds = \frac{1}{2}\sqrt{\gamma}\, b\, U_t.$$

Hence we obtain the normal ordered form of the equation for $U_t$:

$$\frac{d}{dt}U_t = \sqrt{\gamma}\, \mathsf{a}_t^\dagger\, b\, U_t - \sqrt{\gamma}\, b^\dagger\, U_t\, \mathsf{a}_t - \frac{1}{2}\gamma\, b^\dagger b\, U_t - iH\, U_t.$$

This is the white noise form of the quantum Itô Eq. (2), where one can formally write

$$A_t = \int_0^t \mathsf{a}_s\, ds, \quad \int_0^t X_s\, dA_s = \int_0^t X_s\, \mathsf{a}_s\, ds, \quad \int_0^t X_s\, dA_s^\dagger = \int_0^t \mathsf{a}_s^\dagger\, X_s\, ds.$$

To make these ideas rigorous, one would not use white noise. Instead, one should first introduce a mathematically well-posed definition of quantum Itô integrals; this was done by Hudson and Parthasarathy.[16] Then, rather than setting $\gamma(\omega) = \gamma$, one can show that the quantum Itô Eq. (2) is obtained from the non-Markov model above (with suitably regular $\gamma(\omega)$) in a well defined limit, see Refs. 1, 2, 14. Alternatively, Eq. (2) defines a perfectly respectable phenomenological Markov model for the interaction of the atom-cavity system with the external electromagnetic field; for the purposes of this article, it is the starting point for further developments.

We will now show how one of our main results can be reproduced by a formal calculation which is similar to the naive approach used in Ref. 7 (as described in the introduction), but with some crucial corrections. Let us first elaborate a little on the motivation behind this procedure. Roughly speaking, adiabatic elimination problems deal with coupled equations of the form

$$\dot{X}_t = f(X_t, Y_t), \quad \dot{Y}_t = g(X_t, Y_t),$$

where $X$ is a slow variable that we wish to retain, and $Y$ is a fast variable that we wish to eliminate. A well known heuristic, which is widely used (and abused) in the literature, is to argue that the fast variables relax to an equilibrium value so quickly that one can effectively set $\dot{Y}_t = 0$. The technique proceeds by solving the algebraic equation $0 = g(X, Y)$ for $Y$ as a function of $X$, and this expression is then substituted into the equation for $X$ to obtain the adiabatically eliminated equation. One says that the fast variable $Y$ is "slaved" to the slow variable $X$. The motivation for this procedure stems from the fact that for ordinary differential equations, this result can be obtained in certain cases through a rigorous limiting procedure (Tikhonov's theorem[(23)]). However, there is no particular reason to believe *a priori* that this is a sensible thing to do, particularly in the delicate stochastic setting. The formal calculation which we are about to perform is justified only by the fact that it happens to give the right answer (Theorem 2).

Let $X$ be an observable of the atom, and consider the Heisenberg evolution $X_t = U_t^\dagger X U_t$. Using the above white noise form Schrödinger equation for $U_t$, we could write

$$\dot{X}_t = -ib_t^\dagger[X_t, E_{11}(t)]b_t - ib_t^\dagger[X_t, E_{10}(t)] - i[X_t, E_{01}(t)]b_t - i[X_t, E_{00}(t)],$$
$$E_{ij}(t) = U_t^\dagger E_{ij} U_t,$$

where the Heisenberg evolution of the cavity operator $b_t = U_t^\dagger b U_t$ is given by

$$\dot{b}_t = -\sqrt{\gamma}\,\mathsf{a}_t - \frac{\gamma}{2}\,b_t - iE_{11}(t)\,b_t - iE_{10}(t).$$

We consider $X_t$ to be the slow variable, and $b_t$ to be the fast variable which we wish to eliminate. Setting $\dot{b}_t = 0$ allows us to calculate formally a "slaved" form of $b_t$:

$$b_t^{\mathrm{sl}} = -\left(\frac{\gamma}{2} + iE_{11}(t)\right)^{-1}(\sqrt{\gamma}\,\mathsf{a}_t + iE_{10}(t)).$$

Substituting into the equation for $X_t$, we obtain after straightforward but tedious manipulations

$$\begin{aligned}\dot{X}_t &= -ib_t^{\mathrm{sl}\dagger}[X_t, E_{11}(t)]b_t^{\mathrm{sl}} - ib_t^{\mathrm{sl}\dagger}[X_t, E_{10}(t)] - i[X_t, E_{01}(t)]b_t^{\mathrm{sl}} - i[X_t, E_{00}(t)] \\ &= \mathsf{a}_t^\dagger\{\tilde{W}^\dagger(t)X_t\tilde{W}(t) - X_t\}\mathsf{a}_t + [\tilde{L}^\dagger(t), X_t]\tilde{W}(t)\mathsf{a}_t + \mathsf{a}_t^\dagger\tilde{W}^\dagger(t)[X_t, \tilde{L}(t)] \\ &\quad + \tilde{L}^\dagger(t)X_t\tilde{L}(t) - \tfrac{1}{2}\tilde{L}^\dagger(t)\tilde{L}(t)X_t - \tfrac{1}{2}X_t\tilde{L}^\dagger(t)\tilde{L}(t) + i[\tilde{H}(t), X_t].\end{aligned}$$

As this equation is already in normal order, we intepret it as an Itô equation: in particular, this equation corresponds formally to the Hudson–Parthasarathy form

$$\begin{aligned}dX_t &= \{\tilde{W}^\dagger(t)X_t\tilde{W}(t) - X_t\}\,d\Lambda_t + [\tilde{L}^\dagger(t), X_t]\tilde{W}(t)\,dA_t + \tilde{W}^\dagger(t)[X_t, \tilde{L}(t)]\,dA_t^\dagger \\ &\quad + \{\tilde{L}^\dagger(t)X_t\tilde{L}(t) - \tfrac{1}{2}\tilde{L}^\dagger(t)\tilde{L}(t)X_t - \tfrac{1}{2}X_t\tilde{L}^\dagger(t)\tilde{L}(t) + i[\tilde{H}(t), X_t]\}\,dt,\end{aligned}$$

where we have formally introduced the gauge process and integral as

$$\Lambda_t = \int_0^t \mathsf{a}_s^\dagger\mathsf{a}_s\,ds, \qquad \int_0^t X_s\,d\Lambda_s = \int_0^t \mathsf{a}_s^\dagger\,X_s\,\mathsf{a}_s\,ds.$$

This is precisely the result of Theorem 2, as can be seen by applying the quantum Itô rules to $X_t = \tilde{U}_t^\dagger X\tilde{U}_t$ with $\tilde{U}_t$ as defined in Theorem 1.

The procedure above is highly misleading, however. Consider again the equation

$$\dot{X}_t = -ib_t^\dagger[X_t, E_{11}(t)]b_t - ib_t^\dagger[X_t, E_{10}(t)] - i[X_t, E_{01}(t)]b_t - i[X_t, E_{00}(t)].$$

Note that $b_t^\dagger$ commutes with all the commutators; hence we could just as well have written, e.g.,

$$\dot{X}_t = -i[X_t, E_{11}(t)]b_t^\dagger b_t - i[X_t, E_{10}(t)]b_t^\dagger - i[X_t, E_{01}(t)]b_t - i[X_t, E_{00}(t)].$$

If we subsequently substitute $b_t^{\mathrm{sl}}$ for $b_t$, a different equation is obtained which certainly does not coincide with the correct answer even when transformed to normal order. In itself, the fact that $b_t^{\mathrm{sl}}$ does not obey the same commutation relations as $b_t$ makes any procedure of this kind extremely suspicious; had we not known the correct answer to begin with, there would have been no reason to prefer one operator ordering over another, and we would have obtained a whole family of potential answers (none of which are justified). We hope that the reader is convinced that such a naive approach to adiabatic elimination can not be made plausible; it is only in hindsight, having obtained the adiabatically eliminated equation through a different method, that we can select the operator ordering which happens to lead to the right answer.

The remarkable fact that the authors of Ref. 7 succeed in obtaining the correct answer for their particular model (see Example 1 in Sec. 4) can be traced to a miraculous series of cancellations. These authors proceed essentially as we have done above, by substituting $b_t^{\mathrm{sl}}$ for $b_t$ in the equation for $X_t$. However, they do not interpret the resulting equation as an Itô equation, but decide to interpret it as an "implicit" equation, to be converted to "explicit" form through a procedure outlined by Wiseman.[25] This unjustified procedure is itself a mistaken interpretation of a quantum Stratonovich equation, and is known to give rise to incorrect Markov approximations in the presence of the gauge process (as is evident from the discrepancy between Refs. 25 and 14). To complicate matters further, the procedure in Ref. 7 relies crucially on the fact that for their particular choice of $X$ and $E_{ij}$, the commutator $[X, E_{11}]$ commutes with $E_{11}$. This series of coincidences conspires to give the correct answer at the end of the day though a miraculous cancellation of errors. The miracle only occurs in their particular model, however; their procedure quickly fails if the fortuitous commutation relations are not satisfied, e.g., if $E_{ab}$ are functions of angular momentum operators rather than of position and momentum.

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